

ESSENTIAL SELF-ADJOINTNESS OF DIFFERENTIAL OPERATORS ON NON-COMPACT MANIFOLDS WITH LOW-REGULARITY METRICS

LASHI BANDARA AND HEMANTH SARATCHANDRAN

ABSTRACT. We consider first-order differential operators with locally bounded measurable coefficients on vector bundles with measurable coefficient metrics. Under a mild set of assumptions, we demonstrate the equivalence between the essential self-adjointness of such operators to a negligible boundary property. When the operator possesses higher regularity coefficients, we show that higher powers are essentially self-adjoint if and only if this condition is satisfied. In the case that the low-regularity Riemannian metric induces a complete length space, we demonstrate essential self-adjointness of the operator and its higher powers up to the regularity of its coefficients. We also present applications to Dirac operators on Dirac bundles when the metric is non-smooth.

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1. INTRODUCTION

The problem of determining essential self-adjointness of smooth coefficient first-order differential operators, as well as their powers, is an important and well studied topic. This paper considers similar problems but in the context of non-smooth coefficients. More precisely, we consider first-order symmetric differential operators D , as well as their powers, on smooth vector bundles \mathcal{V} , over smooth, noncompact manifolds \mathcal{M} . We allow for the coefficients of the operator as well as the metrics on the bundle and the manifold to be non-smooth. Our primary focus is to understand the relationship between the regularity of the coefficients of D and the essential self-adjointness of powers of D .

There is a plethora of historical literature surrounding this subject, and therefore, we confine ourselves to presenting only the relevant references to our work. From

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our point of view, it was Gaffney who in [12] made a first significant contribution by establishing the essential self-adjointness of the Hodge-Laplacian ($d\delta + \delta d$) under a so-called *negligible boundary* condition. Moreover, Gaffney allows his manifold to be C^k and incomplete. The next relevant reference to us is the work of Wolf in [18], where he studies the essential self-adjointness of general Dirac operators and their squares. Moreover, Cordes in [11] obtains the essential self-adjointness of all powers of the Laplace-Beltrami operator Δ^m on functions. Later, in [9], Chernoff studies conditions under which essential self-adjointness of *all* powers of first-order operators are obtained. These last three references assume both completeness and smoothness of their metrics and the coefficients of the operators.

The study we present here is in part motivated by an attempt to recover results similar to that of [11] and [9] but without assuming smoothness of the operator and the metric. A priori, our differential operator $D : C^{0,1}(\mathcal{V}) \rightarrow L_{\text{loc}}^\infty(\mathcal{V})$, and such operators can arise, for instance, as operators built from Levi-Civita connections associated to Lipschitz metrics. Under a set of mild assumptions (A1)-(A2), we proceed by demonstrating that the essential self-adjointness of this operator is equivalent to the *negligible boundary* property formulated with respect to the operator D (see Definition 2.3). This property is trivially satisfied on compact manifolds and hence, our analysis is exclusively carried out in the non-compact setting. If the differential operator has C^m -coefficients, then for $l \leq m + 1$, we demonstrate that the essential self-adjointness of D^l is equivalent to this aforementioned property for the operator D^l . Indeed, we do not expect essential self-adjointness, in general, to survive for orders $l > m + 1$ for a C^m -coefficient operator. This is partly due to the fact that, even for an arbitrary smooth compactly supported u , $D^l u$ exists only distributionally.

In §4, we demonstrate this equivalence under the mild set of assumptions (A1)-(A2), primarily motivated by features we would expect from elliptic operators. We emphasise that in demonstrating this equivalence, we do not make any assumptions about the completeness of the underlying space. We proceed abstractly in order to emphasise this point, and in fact, carry out this analysis on spaces with so-called *rough metrics*. These are Riemannian-like metrics that have merely measurable coefficients, which are comparable against induced Euclidean metrics in small enough charts. A priori, such a metric only induces a well defined measure, and it is unknown whether there is a naturally associated notion of length. The significance of the negligible boundary criterion is that it can still be formulated on this measure space without alluding to a length structure.

In §5, at this same level of generality, we are also able to show that if D^l is essentially self-adjoint, then so are its lower powers. This is done via functional calculus and operator theoretic methods to emphasise the fact that this assertion makes no geometric demands.

Geometry begins to play a significant role when attempting to boost essential self-adjointness from lower to higher powers. We carry out this study in §6, and the way in which we do this is to establish the negligible boundary condition for higher powers via a bootstrapping procedure. Here, we are forced to assume completeness, but we are able to allow for a restricted class of rough metrics that induce a length

space. This is still a large and significant class of non-trivial metrics which, for instance, include bi-Lipschitz pullbacks of smooth, complete metrics. We emphasise that it is the lack of regularity that forces us away from the wave technique in [9] or the PDE technique in [11], as these techniques seem to require smoothness in a crucial way. By generalising the ideas of [18], we demonstrate a certain elliptic estimate for lower powers of our operator from knowing that the higher powers have finite L^2 -energy. This in turn allows us to obtain negligible boundary for higher powers via the first power.

In §3, we consider applications to Dirac and other elliptic differential operators. As a first consequence, we highlight the following result for Spin manifolds with non-smooth metrics.

Theorem 1.1. *Let g be a $C^{0,1}$ complete metric on a smooth Spin manifold \mathcal{M} , with a spin structure $P_{\text{Spin}}(\mathcal{M})$. Let $\Delta \mathcal{M} = P_{\text{Spin}} \times_{\eta} \Delta \mathbb{R}^n$, where $\eta : \text{Spin}_n \rightarrow \mathcal{L}(\Delta \mathbb{R}^n)$ is the minimal complex irreducible representation. Then, the associated Atiyah-Singer Dirac operator \mathbb{D} is essentially self-adjoint on $C_c^\infty(\mathcal{V})$. If the metric is C^m for $m \geq 1$, then \mathbb{D}^l is essentially self-adjoint on $C_c^\infty(\mathcal{V})$ for $l \leq m$.*

The first part of this theorem, the essential self-adjointness of \mathbb{D} for Lipschitz metrics, was obtained by Bandara, McIntosh and Rosén in [4] by other methods, and was a guiding motivation for our work here. Moreover, Theorem 1.1 is a particular instance of more general results we obtain for Dirac operators on general Dirac bundles.

As a second highlight theorem, we present the following consequence of our work to general symmetric elliptic operators.

Theorem 1.2. *Let \mathcal{V} be a smooth bundle over a smooth manifold \mathcal{M} with continuous metrics h and g . Suppose that D is a first-order elliptic differential operator with C^m -coefficients, $m \geq 0$, that is symmetric on C_c^∞ in $L^2(\mathcal{V})$. Then, D^k on $C_c^\infty(\mathcal{V})$ for $k = 1, \dots, m+1$ is essentially self-adjoint.*

Our theorems generalise the consequences of the work of Wolf in [18] and Chernoff in [9] to Dirac operators to settings where smoothness assumptions on the metrics are relaxed. In Theorem 1.2, we recover the results of Chernoff from [9] in the elliptic setting, and we also dispense with the *local velocity of propagation* condition assumed in that paper.

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2. SETUP AND MAIN RESULTS

2.1. Notation. Throughout this article, we will use the analysts inequality $a \lesssim b$ to mean that $a \leq Cb$ with $C > 0$, as well as the analysts equivalence $a \simeq b$. The support of a section (or function) f will be denoted by $\text{spt } f$. Unless otherwise stated, we will assume Einstein summation convention throughout this paper. That is, whenever a raised index appears multiplicatively against a lowered index, we assume summation over that index. By *precompact set*, we mean a set whose closure is compact.

2.2. Vector bundles over manifolds. Let \mathcal{M} be a connected, non-compact, smooth manifold and $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{M}$ be a smooth vector bundle of finite dimension $\dim \mathcal{V} = N$. By \mathcal{V}_x , we denote the fibre over the point $x \in \mathcal{M}$, given by $\pi_{\mathcal{V}}^{-1}\{x\}$.

The locally Euclidean structure allows us to define spaces of regularity in a local sense. The spaces $C^{k,\alpha}(\mathcal{V})$ will be used to denote k times differentiable sections for which the k -th derivative is α -Hölder continuous *locally*. The notation $C_c^{k,\alpha}(\mathcal{V})$ will be such sections that are compactly supported.

Moreover, this locally Euclidean structure allows us to import measure-theoretic notions in the absence of a metric: we say that a set A is measurable if $\varphi(A \cap U)$ is Lebesgue measurable for all coordinate charts (U, φ) with $U \cap A \neq \emptyset$. This allows us to define measurable functions $f : \mathcal{M} \rightarrow \mathbb{C}$, and by $\Gamma(\mathcal{V})$, we denote measurable sections over \mathcal{V} to be sections $v = v_i e^i$ in continuous local frames $\{e^i\}$ with v_i a measurable function. See [3] for a detailed construction.

Using our notion of measurability, we define $L_{\text{loc}}^1(\mathcal{V})$ to be measurable sections v such that $\psi^{-1}v \in L_{\text{loc}}^1(U, \mathbb{C}^N)$ over local trivialisations $\psi : U \times \mathbb{C}^N \rightarrow \mathcal{V}$. The definition of local Sobolev spaces follow similarly: we say that $u \in W_{\text{loc}}^{k,p}(\mathcal{V})$ if $\psi^{-1}u \in W_{\text{loc}}^{k,p}(U, \mathcal{V})$.

Since our goal is to study *global* differential operators over \mathcal{V} , on letting \mathcal{V}^* be the dual bundle of \mathcal{V} , we define the following notion of a metric tensor.

Definition 2.1 (Rough metrics and bundle rough metrics). *Let $h \in \Gamma(\mathcal{V}^* \otimes \mathcal{V}^*)$ be real and symmetric, and suppose that for each $x \in \mathcal{M}$, there exists a trivialisation (U_x, ψ_x) containing x and a constant $C = C(U_x) \geq 1$ satisfying:*

$$C^{-1} |u|_{h(y)} \leq |\psi_x(y)u|_{\delta} \leq C |u|_{h(y)}$$

for almost-every $y \in U_x$ and $u \in \mathcal{V}_x$, where δ denotes the Euclidean metric in \mathbb{C}^N . Then, we say that such a metric is a bundle rough metric, and such a trivialisation is said to satisfy the local comparability condition.

In the situation where $\mathcal{V} = T\mathcal{M}$, we say that g is a rough metric on \mathcal{M} if in addition, it is real-valued and the trivialisations are induced by coordinate charts.

We distinguish the latter notion of a rough metric from a bundle rough metric since the chart induced trivialisations are necessary to setup a measure. More precisely,

given a rough metric g on \mathcal{M} , we obtain a global measure μ_g by pasting together

$$d\mu_g = \sqrt{\det g_{ij}} d\mathcal{L},$$

via a smooth partition of unity subordinate to a covering of \mathcal{M} by locally comparable charts. This is readily checked to define a Borel-measure that is finite on compact sets. We emphasise that a priori, we do not know whether g induces a length structure. Moreover, a set A is measurable if and only if it is μ_g -measurable.

The notion of rough metrics arose from the study of the geometric invariances of the Kato square root problem. There is a large non-trivial class of such metrics that arise as bi-Lipschitz pullbacks of smooth metrics. A detailed exposition of the properties of rough metrics can be found in [3].

Throughout the remainder of this paper, unless otherwise specified, we fix a rough metric g on \mathcal{M} and a bundle rough metric h on \mathcal{V} . If \mathcal{V} and h are real, we consider h as a complex-valued inner product by complexifying \mathcal{V} . For an open set U , define $L^p(U, \mathcal{V})$ spaces as the set of measurable distributions ξ such that

$$\|\xi\|_{U,p}^p = \|\xi\|_{L^p(U,\mathcal{V})}^p = \int_U |\xi(x)|_{h(x)}^p d\mu_g(x) < \infty.$$

When $U = \mathcal{M}$, we simply write $L^p(\mathcal{V})$ and $\|\xi\|_p = \|\xi\|_{L^p(\mathcal{M},\mathcal{V})}$. In the special case of $p = 2$, the space $L^2(U, \mathcal{V})$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{L^2(U,\mathcal{V})} = \int_U h_x(u(x), v(x)) d\mu_g(x).$$

The $L^2(\mathcal{V})$ inner product will be denoted by $\langle \cdot, \cdot \rangle$ and the induced norm by $\|\cdot\|$.

It can be readily verified that $L_{\text{loc}}^1(\mathcal{V})$ is the space of measurable ξ such that $\xi \in L^1(K, \mathcal{V})$ for all pre-compact $K \Subset \mathcal{V}$. Furthermore, the local Sobolev spaces $W_{\text{loc}}^{k,p}(\mathcal{V})$ can be characterised by $\xi \in L^p(\mathcal{V})$ such that

$$\sum_{i=1}^k \|\nabla^i \xi\|_{K,p} < \infty$$

for every smooth connection ∇ over pre-compact $K \Subset \mathcal{M}$. Note that any two smooth connections ∇_1 and ∇_2 over a pre-compact K are comparable in the sense that

$$\sum_{j=1}^l \|\nabla_1^j u\|_{K,p} + \|u\|_{K,p} \simeq \sum_{j=1}^l \|\nabla_2^j u\|_{K,p} + \|u\|_{K,p}$$

for $l \geq 1$.

2.3. Operator theory. In what is to follow, we will require some notions from operator theory. Fixing a Hilbert space \mathcal{H} , we consider operators $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$, where $\mathcal{D}(T)$ is a subspace of \mathcal{H} called the *domain* of the operator. The *range* of T is denoted by $\mathcal{R}(T)$ and its *null space*, or *kernel*, by $\mathcal{N}(T)$.

An operator is *densely-defined* if $\mathcal{D}(T)$ is dense in \mathcal{H} , and it is *bounded* if there exists a $C > 0$ such that $\|Tu\| \leq C\|u\|$. We say that an operator is *closed* if $u_n \rightarrow u$

and $Tu_n \rightarrow v$ implies that $u \in \mathcal{D}(T)$ and $v = Tu$. This is equivalent to saying that the graph $\mathcal{G}(T) = \{(u, Tu) : u \in \mathcal{D}(T)\}$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. A subspace $\mathcal{B} \subset \mathcal{D}(T)$ is called a *core* for T if it is dense in $\mathcal{D}(T)$ with respect to the graph norm $\|\cdot\|_T = \|\cdot\| + \|T\cdot\|$.

An operator is *closable* if $u_n \rightarrow 0$ and $Tu_n \rightarrow v$ implies that $v = 0$. In that case, $\overline{\mathcal{G}(T)} = \mathcal{G}(\tilde{T})$ where \tilde{T} is a closed operator. We write the closure of the operator as $\overline{T} = \tilde{T}$. A densely-defined operator T admits a closed operator T^* called *the adjoint* of T with domain

$$\mathcal{D}(T^*) = \{u \in \mathcal{H} : v \mapsto \langle Tv, u \rangle \text{ is continuous}\}.$$

The operator T^* is defined as follows: for $u \in \mathcal{D}(T^*)$, there exists $f_u \in \mathcal{H}$ such that $\langle Tv, u \rangle = \langle v, f_u \rangle$ by the Riesz-representation theorem, and $T^*u = f_u$.

An operator S is said to be *an adjoint* of T if $\langle Tu, v \rangle = \langle u, Sv \rangle$ for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(S)$. In particular, whenever T and S are densely-defined, they are both closable and admit densely-defined adjoints. An operator T is symmetric if T is *an adjoint* to itself. However, typically $T \subset T^*$, by which we mean that $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T^* = T$ on $\mathcal{D}(T)$. An operator T is *self-adjoint* if $T^* = T$. A symmetric operator is *essentially self-adjoint* if it admits a unique self-adjoint extension T_s . In this situation, it is readily checked that $\overline{T} = T_s = T^*$.

2.4. Main results. Throughout this paper, whenever we say *first-order differential operator*, we will assume it is an operator that takes the form $A^i \partial_i + B$ locally, with coefficients $A^i, B \in L_{\text{loc}}^\infty(\mathcal{V}^* \otimes \mathcal{V})$, and $A^i \neq 0$ for some i . This is a linear map $D : C^{0,1}(\mathcal{V}) \rightarrow L_{\text{loc}}^\infty(\mathcal{V})$, that is local and for which $M_\eta = [D, \eta I]$ is an almost-everywhere nonzero, fibrewise, bounded multiplication operator for $\eta \in C^{0,1}(\mathcal{M})$. Note that for almost-every $x \in \mathcal{M}$ and each fibre norm $|\cdot|_x$ on \mathcal{V}_x , there exist $C_x > 0$ such that $|M_\eta(x)| \leq C_x |\nabla \eta(x)|$.

If for $m \geq 0$, $D : C^\infty(\mathcal{V}) \rightarrow C^m(\mathcal{V})$, and $D : C^l(\mathcal{V}) \rightarrow C^{l-1}(\mathcal{V})$ for $0 < l \leq m+1$, then we say that D has C^m coefficients.

Throughout, let $l \leq m+1$ for a C^m coefficient operator (or $m=0$ if the operator has L_{loc}^∞ coefficients) and define:

$$\mathcal{S}_l = \mathcal{S}_l(D) = \{u \in C^\infty \cap L^2(\mathcal{V}) : D^l u \in L^2\}.$$

Denote the l -graph norm of the operator by: $\|u\|_{D^l} = \|D^l u\| + \|u\|$. For a function $\eta \in C^\infty(\mathcal{M})$, define the commutators $M_\eta^l u = [D^l, \eta I] u$, for $u \in C^\infty(\mathcal{V})$. The commutator M_η^1 will be denoted by M_η .

We present the following two axioms under which we prove the most general results of this paper. In §3, we illustrate that these axioms are valid for a wide class of elliptic operators.

- (A1) whenever $u \in \mathcal{S}_1(D)$ and $v \in C_c^\infty(\mathcal{V})$, $\langle Du, v \rangle = \langle u, Dv \rangle$,
- (A2) whenever $u \in L^2(\mathcal{V})$ and $D^l u \in L_{\text{loc}}^2(\mathcal{V})$, then $u \in W_{\text{loc}}^{l,2}(\mathcal{V})$.

Remark 2.2. The condition (A2) is an L^2 -ellipticity condition on the operator D . It is automatically satisfied for C^m coefficient elliptic operators. We formulate this as an axiom since we only require this weaker formulation, and as we shall see in §3, it can be proved in instances where the coefficients are merely L^∞_{loc} .

For integers $1 \leq k \leq m+1$, let $D_c^k = D^k$ with domain $\mathcal{D}(D_c^k) = C_c^\infty(\mathcal{V})$ and $D_2^k = D^k$ with domain $\mathcal{D}(D_2^k) = \mathcal{S}_k(D)$. For the case $k=1$, (A1) implies that D_2 and D_c are closable, and on letting $D_N = \overline{D_2}$ and $D_D = \overline{D_c}$, we obtain that $\langle D_N u, v \rangle = \langle u, D_D v \rangle$ for $u \in \mathcal{D}(D_N)$ and $v \in \mathcal{D}(D_D)$. Moreover, using (A2) and a mollification argument, we obtain that $C_c^{0,1}(\mathcal{V}) \subset \mathcal{D}(D_D)$. Fix $v \in C_c^\infty(\mathcal{V})$, let V be a pre-compact set satisfying $\text{spt } v \subset V$, and $\{\eta_i\}_{i=1}^M$ be smooth partition of unity on V . For $u \in \mathcal{S}_k(D)$, write $u = \sum_{i=1}^M \eta_i u$ on V . Set $u_i = \eta_i u$, extended to zero outside of $\text{spt } \eta_i$. Since D is a C^m coefficient operator, we have that $D^p(u_i) \in C_c^{m-p}(\mathcal{V}) \subset C_c^{0,1}(\mathcal{V})$ for $p \leq k-1$. Then,

$$\begin{aligned} \langle D^k u, v \rangle &= \sum_{i=1}^M \langle D_N^k(u_i), v \rangle = \sum_{i=1}^M \langle D_N D_N^{k-1}(u_i), v \rangle \\ &= \sum_{i=1}^M \langle D_N^{k-1} u_i, D_D v \rangle = \cdots = \sum_{i=1}^M \langle u_i, D_D^k v \rangle = \langle u, D^k v \rangle. \end{aligned}$$

That is,

$$\langle D^k u, v \rangle = \langle u, D^k v \rangle, \quad \forall u \in \mathcal{S}_k(D), \quad \forall v \in C_c^\infty(\mathcal{V}).$$

Observe that if we simply asked that D be symmetric on $C_c^\infty(\mathcal{V})$, then a similar calculation would yield (A1) as a consequence.

As a consequence of this symmetry condition for D^k , we obtain that the operators D_c^k and D_2^k are closable and as for the $k=1$ case, we write $(D^k)_D$ and $(D^k)_N$ respectively to denote the closures of these operators with domains $\mathcal{D}_0^k(D)$ and $\mathcal{D}^k(D)$. By viewing D^k distributionally, i.e., for $u \in L^2(\mathcal{V})$ defining $(D^k u)(v) = \langle u, D^k v \rangle$ for $v \in C_c^\infty(\mathcal{V})$, define the maximal domain as

$$\mathcal{D}_\infty^k(D) = \{u \in L^2(\mathcal{V}) : D^k u \in L^2(\mathcal{V})\}.$$

In addition to the conditions (A1)-(A2), a fundamental criteria we use and exploit throughout this paper is the following.

Definition 2.3 (Negligible Boundary). *We say that the operator D^l exhibits negligible boundary if*

$$\langle D^l u, v \rangle = \langle u, D^l v \rangle \quad (l\text{-Neg})$$

for all $u, v \in \mathcal{S}_l(D)$.

The first theorem we present is the following. It is proved in §4.

Theorem 2.4. *Let D^l satisfy (A1)-(A2) on a bundle \mathcal{V} with a bundle rough metric h over \mathcal{M} with a rough metric g . Then, the following are equivalent:*

$$(i) \quad \mathcal{D}_0^l(D) = \mathcal{D}^l(D),$$

- (ii) $(D^l)_D$ is self-adjoint,
- (iii) $(D^l)_N$ is self-adjoint,
- (iv) D^l satisfies **(l-Neg)**,
- (v) D_c^l is essentially self-adjoint.

The next theorem allows us to deduce essential self-adjointness of lower powers of the operator when we know this for higher powers. It is proved in §5.

Theorem 2.5. *Let D satisfy **(A1)**-**(A2)** on a bundle \mathcal{V} with a bundle rough metric h over \mathcal{M} with a rough metric g . If D^l on $C_c^\infty(\mathcal{V})$ is essentially self-adjoint, then $C_c^\infty(\mathcal{V})$ is a core for $|D_D|^\alpha$ for $\alpha \in [0, k]$ and moreover, it is a core for D_D^k for $k = 1, \dots, l$.*

Remark 2.6. *This result requires very little geometric properties: the theorem is phrased for rough metrics which may not, a priori, even induce a length structure. In particular, it does not require completeness.*

Moreover, note that we only require the first power of D to satisfy **(A1)**-**(A2)** in this theorem.

The second theorem we present is in the case that the rough metric induces a length structure. By this, we mean that

$$\rho(x, y) = \inf \{ \ell(\gamma) : \gamma \text{ is an absolutely continuous curve between } x \text{ and } y \},$$

where $\ell(\gamma) = \int_I |\dot{\gamma}(x)|_{g(x)}$, is a distance metric which yields a topology that agrees with the topology on \mathcal{M} . The quintessential nontrivial example of such a situation is the pullback metric $g = \varphi^* \tilde{g}$, where \tilde{g} is a smooth metric and φ is a lipeomorphism. The metric g , in general, only has measurable coefficients, and φ can be seen as a bi-Lipschitz transformation of \tilde{g} .

Theorem 2.7. *Let \mathcal{V} be a vector bundle with a bundle rough metric h , over a manifold \mathcal{M} with a rough metric g that induces a complete length space. If D is a first-order operator satisfying **(A1)**-**(A2)**, then D on $C_c^\infty(\mathcal{V})$ is essentially self-adjoint. If D has C^m coefficients and D^k satisfies **(A1)**-**(A2)** for $1 \leq k \leq m+1$, then D^k on $C_c^\infty(\mathcal{V})$ is essentially self-adjoint for $1 \leq k \leq m+1$.*

The proof of this theorem can be found in §6.3.

Remark 2.8. *In [18], Wolf obtains essential self-adjointness for D and D^2 when D is the Dirac operator for smooth complete metrics. Following from this, Chernoff in [9] uses wave techniques to obtain such a result for all powers D^k of first-order differential operators satisfying a velocity of propagation condition. The wave techniques crucially rely on smoothness.*

In our theorem, we only assume completeness and allow for bad singularities at the level of the metric. We keep track of the way in which the coefficients of our differential operator affects the density of $C_c^\infty(\mathcal{V})$ for higher powers.

3. APPLICATIONS

3.1. Dirac operators on Dirac bundles. Fix a $C^{0,1}$ metric g on \mathcal{M} . By the usual formula for the Christoffel symbols, we define the Levi-Civita connection $\nabla : C^{0,1}(T\mathcal{M}) \rightarrow L_{\text{loc}}^\infty(T^*\mathcal{M} \otimes T\mathcal{M})$.

Let $\Delta\mathcal{M}$ denote the Clifford algebra, which is the exterior algebra $\Omega\mathcal{M}$ equipped with the Clifford product \triangle given by $\alpha \triangle \beta = \alpha \wedge \beta + \alpha \lrcorner \beta$. By the symbol ∇ , we denote the canonical lift of the Levi-Civita connection to the bundle $\Delta\mathcal{M}$.

Following the notation of [14], we say that a complex bundle \mathcal{S} is a Spin bundle over (\mathcal{M}, g) if:

- (i) it is a bundle of left-modules over $\Delta\mathcal{M}$ with $\cdot : \Delta\mathcal{M} \times \mathcal{S} \rightarrow \mathcal{S}$,
- (ii) it is equipped with a hermitian $C^{0,1}$ -metric and a compatible connection $\nabla : C^{0,1}(\mathcal{S}) \rightarrow L_{\text{loc}}^\infty(T^*\mathcal{M} \otimes \mathcal{S})$ satisfying:

$$\begin{aligned} h_x(e \cdot \sigma, e \cdot \gamma) &= h_x(\sigma, \gamma), \text{ and } \nabla_v(\eta \cdot u) = (\nabla_v \eta) \cdot u + \eta \cdot \nabla_v u \\ \text{for all } |e| &= 1 \text{ in } T_x^*\mathcal{M}, v \in T\mathcal{M} \text{ and for almost-every } x \in \mathcal{M}. \end{aligned} \quad (\text{S})$$

The first property says that \cdot is unitary and the second property says that ∇ is a module derivation.

Fixing a frame $\{e_i\}$ for $T\mathcal{M}$, recall that the Dirac operator D is defined by

$$D\sigma = e^i \cdot \nabla_{e_i} \sigma.$$

It is easy to check that this is a first-order differential operator, $D : C^{0,1}(\mathcal{S}) \rightarrow L_{\text{loc}}^\infty(\mathcal{S})$ as a consequence of the module derivation property in (S).

We now prove that D satisfies (A1). But first, we present the following lemma that will be the primary tool used in its proof.

Lemma 3.1. *Let $f \in C_c^{0,1}(T^*\mathcal{M})$, a compactly supported Lipschitz co-vectorfield. Then,*

$$\int_{\mathcal{M}} \text{div}_g f \, d\mu_g = 0.$$

Proof. Since the metric g is at least continuous, we note as in Proposition 13 of [3] that, given any $C > 1$, there is a smooth metric \tilde{g} which is C -close to g , by which we mean that: $C^{-1}|u|_{\tilde{g}} \leq |u|_g \leq C|u|_{\tilde{g}}$ for every $x \in \mathcal{M}$ and $u \in T_x\mathcal{M}$.

Note that there exists an $E \in C^{0,1}(\mathcal{T}^{(1,1)}\mathcal{M})$ symmetric such that $g(u, v) = \tilde{g}(Eu, v)$, and on letting $\theta = \sqrt{\det E}$, we have that $\mu_g = \theta\mu_{\tilde{g}}$. Also, E is bounded with respect to both metrics g and \tilde{g} . Computing in L^2 allows us to assert that $\text{div}_g f = \theta^{-1} \text{div}_{\tilde{g}}(\theta E f)$ (c.f. Proposition 12 in [3]). Therefore,

$$\int_{\mathcal{M}} \text{div}_g f \, d\mu_g = \int_{\mathcal{M}} \theta^{-1} \text{div}_{\tilde{g}}(\theta E f) \, d\mu_g = \int_{\mathcal{M}} \text{div}_{\tilde{g}}(\theta E f) \, d\mu_{\tilde{g}}.$$

But since the metric \tilde{g} is smooth, $\theta E \in C^{0,1}(\mathcal{T}^{(1,1)}\mathcal{M})$, $f \in C_c^{0,1}(T^*\mathcal{M})$, and therefore $\theta E f \in C_c^{0,1}(T^*\mathcal{M})$. Thus, by Lemma 7.113 in [10], we obtain that $\int_{\mathcal{M}} \operatorname{div}_{\tilde{g}}(\theta E f) d\mu_{\tilde{g}} = 0$. \square

With this, we prove the following.

Proposition 3.2. *Whenever $\sigma, \gamma \in C_c^\infty(\mathcal{S})$, we have that $\langle D\sigma, \gamma \rangle = \langle \sigma, D\gamma \rangle$. Moreover, D satisfies (A1).*

Proof. Fix a point $x \in \mathcal{M}$ where g and h are differentiable and fix a smooth frame $\{e_j\}$ near x . Let $\nabla_{e_j} e_k = C_{jk}^m e_m$ and note that by the lifting of ∇ to $T^*\mathcal{M}$, we have that $\nabla_{e_j} e^k = -C_{ji}^k e^i$. Using metric compatibility, we obtain

$$\begin{aligned} h(D\sigma, \gamma) &= h(e^j \cdot \nabla_{e_j} \sigma, \gamma) = -h(\nabla_{e_j} \sigma, e^j \cdot \gamma) \\ &= h(\sigma, \nabla_{e_j} (e^j \cdot \gamma)) - e_j h(\sigma, e^j \cdot \gamma) \\ &= -C_{jm}^k h(\sigma, e^m \cdot \gamma) \delta_k^j - e_j h(\sigma, e^j \cdot \gamma) + h(\sigma, D\gamma). \end{aligned}$$

Now, note that for $w = w_i e^i \in C^{0,1}(T\mathcal{M})$,

$$-\operatorname{div} w = \operatorname{tr} \nabla w = e_j w^j + w^j \operatorname{tr}(e^j \otimes \nabla_{e_j} e_i) = e_j w^j + w^i C_{ij}^k \delta_k^j.$$

Define the vectorfield $W_{\sigma, \gamma}(x) = h_x(\sigma(x), e^k(x) \gamma(x)) e_k(x)$, extended to the whole of \mathcal{M} by zero outside of $\operatorname{spt} \sigma \cap \operatorname{spt} \gamma$. It is easy to see that this is invariantly defined and that it is $C^{0,1}$. Thus, from combining these calculations, we see that $h(D\sigma, \gamma) = \operatorname{div} W_{\sigma, \gamma} + h(\sigma, D\gamma)$ at points of differentiability. Thus, by invoking Lemma 3.1, we obtain that

$$\int_{\mathcal{M}} \operatorname{div} W_{\sigma, \gamma} d\mu_g = 0,$$

and therefore, D is symmetric on $C_c^\infty(\mathcal{S})$. The proof of (A1) follows from this as noted in §2.4. \square

Remark 3.3. *Observe that unlike the smooth case, we cannot assume that we can solve for a frame in which $(\nabla_{e_j} e_i)|_x = 0$, since the metrics g and h are merely $C^{0,1}$.*

Next, fix $v \in T_x \mathcal{M}$ and $u \in \mathcal{S}_x$. Let $\eta : \mathcal{M} \rightarrow \mathbb{R}$ be a compactly supported smooth function such that $d\eta(x) = v$. Since D is a first-order differential operator, the symbol of D at (x, v) is:

$$\operatorname{Sym}_D(x, v)u = M(\eta)u = D(\eta u) - \eta Du = d\eta \cdot u = v \cdot u.$$

The critical regularity case in the following theorem is modelled on the proof of Proposition 3.4 in [4].

Proposition 3.4. *D satisfies (A2). If the metrics g and h are C^m for $m \geq 0$ and D is a C^m coefficient operator, (A2) holds for D^l for $l \leq m + 1$.*

Proof. First, observe that any two smooth connections ∇_1 and ∇_2 are comparable on precompact subsets in L^2 -norm. Hence, to show (A2), it suffices to show that $D\sigma \in L^2(\mathcal{S})$ implies that $\sigma \in W^{1,2}(U, \mathcal{S})$ inside precompact trivialisations (U, φ)

corresponding to charts. Also note that $Du \in L^2(\mathcal{S})$ means exactly that $|\langle D\sigma, \gamma \rangle| = |\langle \sigma, D\gamma \rangle| \lesssim \|\gamma\|$ for all $\gamma \in C_c^\infty(\mathcal{S})$.

Suppose that $\sigma \in L^2(\mathcal{S})$ with $\text{spt } \sigma \subset U$ compact. Fix $\gamma \in C_c^\infty(\mathcal{S})$ with $\text{spt } \gamma \subset U$. Since we are inside a precompact chart and g is $C^{0,1}$, we obtain a constant $C \geq 1$ such that

$$C^{-1} |u|_g \leq |u|_{\varphi^* \delta} \leq C |u|_g$$

for all $x \in U$, where by δ we denote the Euclidean metric. Moreover, there exists a transformation $B \in C^{0,1}(\mathcal{T}^{(1,1)}\mathcal{M})$ such that $g(u, v) = \varphi^* \delta(Bu, v)$ and so that $d\mu_g = \theta d\varphi^* \mathcal{L}$, where $\theta = \sqrt{\det B}$ and $\varphi^* \mathcal{L}$ is the pullback of the Lebesgue measure in $\varphi(U)$ by φ . Then, we obtain by the fact that $D\sigma \in L^2(\mathcal{S})$ that $|\langle \sigma, D\gamma \rangle| \lesssim \|\gamma\|$. Expanding this norm, taking an orthonormal frame $\{e_i\}$ for $T\mathcal{M}$ (note that this frame is Lipschitz) and $\{s_j\}$ for \mathcal{S} , we have that

$$\left| \int_U h(\sigma, e^i \cdot (\nabla_{e_i} \gamma^k) s_k) \theta d\mathcal{L} \right| \lesssim \|\gamma\|$$

since $C_{ij}^k = e^k(\nabla_{e_i} s_j) \in L^\infty(U, \mathcal{S})$. Furthermore, $\nabla_{e_i} \gamma^k \theta = \nabla_{e_i}(\theta \gamma^k) - (\nabla_{e_i} \theta) \gamma^k$, and so again, we have that

$$\left| \int_U h(\sigma, e^i \cdot (\nabla_{e_i} \gamma^k) s_k) d\mathcal{L} \right| \lesssim \|\gamma\|_{L^2(U, \mathcal{S}, \mathcal{L})}.$$

Since we fixed an orthonormal frame s_k , the action $e^i \cdot s_k = \eta(e^i) s_k$ for a matrix $\eta(e^i)$, where the coefficients of $\eta(e^i)$ are constant inside U . Viewing this integral on \mathbb{R}^n and extending σ to the whole of \mathbb{R}^n by setting it to 0 outside of $\varphi(U)$, we obtain for any $\gamma \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$ (where $\mathbb{C}^N \cong \mathcal{S}$ inside U), $|\langle \sigma, \nabla_{e_i}(\theta \gamma^k) e^i \cdot s_k \rangle| \lesssim \|\gamma\|$. Since the matrix $\eta(e^i)$ is constant coefficient, we have that $\widehat{e^i \cdot \psi} = e^i \cdot \widehat{\psi}$ for any $\psi \in L^2(\mathbb{R}^n, \mathbb{C}^N)$, where $\widehat{\cdot}$ denotes the Fourier Transform. Then,

$$\langle \sigma, \nabla_{e_i}(\theta \gamma^k) e^i \cdot s_k \rangle = \left\langle \widehat{\sigma}, e^i \cdot \xi_i(\widehat{\theta \gamma^k}) \right\rangle.$$

This is valid for any $\gamma \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$, therefore, $\xi \cdot \widehat{\sigma} = \xi_i e^i \cdot \widehat{\sigma} \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. Hence, $\sigma \in W^{1,2}(\mathbb{R}^n, \mathbb{C}^N)$ which implies that $\sigma \in W_{\text{loc}}^{1,2}(U, \mathcal{S})$.

For a general $\sigma \in L^2(\mathcal{S})$ with $D\sigma \in L^2(\mathcal{S})$, fix \tilde{U} a precompact chart and $U \Subset \tilde{U}$. There is a smooth partition of unity $\{\rho_j\}$ inside \tilde{U} such that $\eta = \sum_{j=1}^M \rho_j = 1$ on U , and we extend $\eta = 0$ outside \tilde{U} . Then, for any $\gamma \in C_c^\infty(\mathcal{S})$,

$$|D(\eta\sigma)(\gamma)| = \langle \sigma, \eta D\gamma \rangle = \langle \sigma, D(\eta\gamma) \rangle - \langle \sigma, d\eta \cdot \gamma \rangle.$$

Since $D\sigma \in L^2(\mathcal{S})$ we have $|\langle \sigma, D(\eta\gamma) \rangle| \lesssim \|\eta\gamma\| \leq \|\gamma\|$ and it is easy to see, from the Cauchy-Schwarz inequality, that $\langle \sigma, d\eta \cdot \gamma \rangle \lesssim \|\sigma\| \|\gamma\|$ (where the constant contains $\sup |d\eta| \lesssim 1$). This proves that $D(\eta\sigma) \in L^2(\mathcal{S})$, and since $\|D\sigma\|_{L^2(U, \mathcal{S})} \leq \|D(\eta\sigma)\|$, by running our previous argument with $\eta\sigma$ in place of σ , we obtain that $\sigma \in W_{\text{loc}}^{1,2}(\mathcal{S})$ whenever $D\sigma \in L^2(\mathcal{S})$. The estimate in (A2) is immediate. This proves the critical regularity case.

For the case that the coefficients of the metric and operator are C^m , and since $\text{Sym}_D(x, v)\sigma = v \cdot \sigma$ implies that D is an elliptic operator, (A2) follows from the classical Schauder interior regularity estimate. See Theorem 10.3 in [8]. \square

On combining these propositions, we use Theorem 2.4 and Theorem 2.7 to obtain the following theorem. The metric g appearing in this theorem automatically induces a length space as a consequence of Proposition 4.1 in [7] by Burtscher.

Theorem 3.5. *The operator D on $C_c^\infty(\mathcal{S})$ is essentially self-adjoint. If the metric g and h are of class C^m for $m \geq 0$, and D is a C^m coefficient operator, then D^l on $C_c^\infty(\mathcal{S})$ is essentially self-adjoint for $1 \leq l \leq m+1$. Moreover, $C_c^\infty(\mathcal{S})$ is a core for $|D_D|^\beta$ for $\beta \in [0, m+1]$.*

The following are then two noteworthy consequences. We remind the reader that in these corollaries, a C^m coefficient metric induces a C^{m-1} coefficient connection, from which the respective operators are built.

Corollary 3.6. *Let g be a $C^{0,1}$ complete metric on a smooth manifold \mathcal{M} . Then, the Hodge-Dirac operator $D = d + \delta$ on $C_c^\infty(\Omega\mathcal{M})$, where $\Omega\mathcal{M}$ is the exterior-algebra, is essentially self-adjoint. If g is C^m , $m \geq 1$, then we get that D^l on $C_c^\infty(\Omega\mathcal{M})$ is essentially self-adjoint for $1 \leq l \leq m$.*

Remark 3.7. *Gaffney in [12] obtains this result for D^2 on C^m manifolds. Roelcke in [17] obtains this theorem for D^2 restricted to functions, and Cordes [11] obtains this for D^m restricted to functions when the metric is smooth. Gaffney in [12] obtains it for all powers under the smoothness assumption.*

Corollary 3.8 (Theorem 1.1). *Let g be a $C^{0,1}$ complete metric on a smooth Spin manifold \mathcal{M} , with a spin structure $P_{\text{Spin}}(\mathcal{M})$. Let $\not{A}\mathcal{M} = P_{\text{Spin}} \times_\eta \not{A}\mathbb{R}^n$, where $\eta : \text{Spin}_n \rightarrow \mathcal{L}(\not{A}\mathbb{R}^n)$ is the minimal complex irreducible representation. Then, the associated Atiyah-Singer Dirac operator \not{D} is essentially self-adjoint on $C_c^\infty(\mathcal{V})$. If the metric is C^m for $m \geq 1$, then \not{D}^l is essentially self-adjoint on $C_c^\infty(\mathcal{V})$ for $l \leq m$.*

Remark 3.9. *When the metric g is smooth, Wolf in [18] obtains this for \not{D} and \not{D}^2 , and Chernoff in [9] obtains this for all powers \not{D}^m . The $C^{0,1}$ metric case is proved in [4] by Bandara, McIntosh and Rosén.*

3.2. Elliptic operators with C^m coefficients. Here, we consider elliptic first-order differential operators D with C^m coefficients. As a consequence of Theorem 2.7, we obtain the following.

Theorem 3.10. *Let \mathcal{V} be a smooth bundle over a smooth manifold \mathcal{M} with rough metrics h and g . Suppose that g induces a complete length space. Let D be a first-order elliptic differential operator with C^m coefficients, $m \geq 0$, that is symmetric on $C_c^\infty(\mathcal{V})$ in $L^2(\mathcal{V})$. Then, D^k on $C_c^\infty(\mathcal{V})$ for $k = 1, \dots, m+1$ is essentially self-adjoint.*

Proof. Fix $v \in C_c^\infty(\mathcal{V})$ and $u \in \mathcal{S}_1(\mathcal{V})$. By a partition of unity argument, we can find $\varphi \in C_c^\infty(\mathcal{M})$ such that $\varphi = 1$ on $\text{spt } v$. Thus, on $\text{spt } v$, $\varphi u = u$, and hence,

$$\langle Du, v \rangle = \langle D(\varphi u), v \rangle = \langle \varphi u, v \rangle = \langle u, Dv \rangle,$$

where the second equality follows from the symmetry of D on $C_c^\infty(\mathcal{V})$. Thus, (A1) is satisfied. The condition (A2) is a consequence of elliptic regularity, see Theorem 10.3 in [8]. \square

Remark 3.11. *This theorem is a generalisation of [9] by Chernoff for elliptic operators and with minimal regularity assumptions on the underlying metrics. Moreover, Chernoff assumes that $\int_0^\infty c(r)^{-1} dr = +\infty$, where $c(r)$ is the “local velocity of propagation” for D inside a ball of radius $r > 0$. In the elliptic context, we are able to show that such an assumption is not necessary.*

4. ESSENTIAL SELF-ADJOINTNESS AND THE NEGLIGIBLE BOUNDARY CONDITION

In this section, we establish the equivalence of negligible boundary and essential self-adjointness for powers of operators. This is the crucial property we exploit throughout the later parts of this paper. We emphasise that the background geometric assumptions here are minimal. In particular, we do not assume completeness.

The central tool is for us to be able to equate $\mathcal{D}^l(D)$ with $\mathcal{D}_\infty^l(D)$. We establish some preliminary lemmas that will aid us in obtaining this equality.

Lemma 4.1. *Let $u \in W_{\text{loc}}^{l,2}(\mathcal{V})$ with $\text{spt } u$ compact, U a precompact open set with $\text{spt } u \subset U$ and ∇^U any smooth connection in U . Then, $u \in \mathcal{D}_0^l(D)$ and there exists a constant $C > 0$ (dependent on ∇^U and U) such that*

$$\|D^l u\|_{L^2(U,\mathcal{V})} \leq C \left(\sum_{i=1}^l \|(\nabla^U)^i u\|_{L^2(U,\mathcal{V})} + \|u\|_{L^2(U,\mathcal{V})} \right).$$

Proof. Observe that, via a partition of unity argument, we can assume that $\text{spt } u \subset U$ with U corresponding to a chart. Also, since U is precompact, any two smooth connections, as well as all of their powers, are comparable. Thus, we assume that ∇^U is the flat connection inside U with respect to a fixed trivialisation.

Next, note by the hypothesis on u , we have that $u \in W^{l,2}(U, \mathcal{V})$, where by what we have said we can fix the norm $\|u\|_{W^{1,2}(U,\mathcal{V})} = \sum_{j=1}^l \|(\nabla^U)^j u\| + \|u\|$. Thus, there is a sequence $u_n \in C_c^\infty(\mathcal{V})$ such that $u_n \rightarrow u$ in $\|\cdot\|_{W^{1,2}(U,\mathcal{V})}$. Using the fact that $D = A^i \partial_i + B$ inside U , we obtain

$$\|D^l(u_n - u_m)\| \lesssim \sum_{j=1}^l \|(\nabla^U)^j(u_n - u_m)\|_U + \|u_n - u_m\|_U.$$

As $m, n \rightarrow \infty$, the right hand side of this expression tends to zero, and hence, we obtain that $D^l u_n \rightarrow v$. Since D^l is closable and the sequence $u_n \in C_c^\infty(\mathcal{V})$, we conclude that $u \in \mathcal{D}_0^l(D)$ and $v = (D^l)_{D^l} u$. The estimate follows easily. \square

With the aid of this lemma, we compute the maximal domain. Our proof here is inspired by the work of Masamune in [16] (see Theorem 2, p.114).

Proposition 4.2. *Under the assumptions (A1)-(A2) for D^l , we have the equality $\mathcal{D}^l(D) = \mathcal{D}_\infty^l(D) = \mathcal{D}((D_c^l)^*)$.*

Proof. We first prove the second equality. For that, note that:

$$\begin{aligned}\mathcal{D}_\infty^l(\mathcal{D}) &= \{u \in L^2(\mathcal{V}) : D^l u \in L^2(\mathcal{V})\} \\ &= \{u \in L^2(\mathcal{V}) : |(D^l u)(v)| = |\langle u, D^l v \rangle| \lesssim \|v\|, \forall v \in C_c^\infty\} \\ &= \{u \in L^2(\mathcal{V}) : v \mapsto \langle u, D^l v \rangle \text{ is continuous}\} = \mathcal{D}((D_c^l)^*).\end{aligned}$$

To prove the first equality, it suffices to show that $\mathcal{D}_\infty^l(\mathcal{D}) \subset \mathcal{D}^l(\mathcal{D})$. Fix $u \in \mathcal{D}_\infty^l(\mathcal{D})$, let $\{(U_i, \psi_i)\}$ be a pre-compact open cover of \mathcal{M} by locally comparable trivialisations/charts for both \mathcal{M} and \mathcal{V} . Let $\{\eta_i\}$ be a smooth partition of unity subordinate to $\{U_i\}$. On writing $u_i = \eta_i u$, it is clear that $u = \sum_{i=1}^\infty u_i$ pointwise almost-everywhere and also in L^2 .

We show that $u_i \in \mathcal{D}_\infty^l(\mathcal{D})$. For that, observe that by (A2), we obtain $u \in W_{\text{loc}}^{l,2}(\mathcal{V})$. Fix a choice of smooth frame $\{e_{i,j}\}_{j=1}^N$ in U_i , and the flat connection ∇ with respect to $\{e_{i,j}\}$ inside U_i . Also, we have

$$\nabla^l(\eta_i u) = \sum_{j=0}^l C_l^j \nabla^j \eta_i \otimes \nabla^{l-j} u,$$

where $C_l^j = \frac{l!}{j!(l-j)!}$. Since η_i is smooth, $\nabla^j \eta_i$ is bounded inside U_i for all $j > 0$, by considering a sequence $u_i^n \rightarrow u_i$ in $\|\cdot\|_{W^{1,2}(U_i, \mathcal{V})}$ with $u_i^n \in C^\infty(U_i, \mathcal{V})$, we deduce that $u_i \in W^{1,2}(U_i, \mathcal{V})$. Therefore, by Lemma 4.1, we obtain that $u_i \in \mathcal{D}_0^l(\mathcal{D}) \subset \mathcal{D}_\infty^l(\mathcal{D})$.

Define $u_i^\varepsilon = (J_\varepsilon * u_i^j) e_{i,j}$, for $\varepsilon > 0$, where J_ε is the standard symmetric mollifier. We can choose $\varepsilon < \varepsilon_i$ so that $\text{spt } u_i^\varepsilon \subset U_i$. By Lemma 4.1, there is a constant C_i such that

$$\|D^l u_i - D^l u_i^\varepsilon\| \leq C_i \left(\sum_{j=1}^l \|\nabla^j (u_i - u_i^\varepsilon)\|_{L^2(U_i, \mathcal{V})} + \|u_i - u_i^\varepsilon\|_{L^2(U_i, \mathcal{V})} \right).$$

Since we assume that U_i satisfy the local comparability condition, setting $f_i = u_i - u_i^\varepsilon$, we have

$$\|\nabla^j f_i\|_{L^2(U_i, \mathcal{V})}^2 = \int_{U_i} |\nabla^j f_i(x)|_{(\mathfrak{g} \otimes \mathfrak{h})(x)}^2 d\mu_g \lesssim \int_{U_i} |\nabla^j f_i(x)|_{\delta(x)}^2 d\psi_i^* \mathcal{L}(x),$$

where $\psi_i^* \mathcal{L}$ is the pullback of the Lebesgue measure inside U_i . By standard mollification theory, we have that the right hand side of this expression tends to zero as $\varepsilon \rightarrow 0$. Also, observe that $u_i^\varepsilon \in \mathcal{S}_l(\mathcal{D})$.

Fix $\delta > 0$ and for each i , choose $\delta_i > 0$ such that

$$\|D^l u_i - D^l u_i^{\delta_i}\| \leq \frac{\delta}{2^i}.$$

For each $x \in \mathcal{M}$, define $u^\delta(x) = \sum_{i=1}^\infty u_i^{\delta_i}(x)$, this is well-defined because $\{\eta_i\}$ is locally finite. It is easy to see that $u^\delta \in C^\infty(\mathcal{V})$, and to show that $u^\delta \in \mathcal{S}_l(\mathcal{D})$, it suffices to show that $D^l u^\delta \in L^2(\mathcal{V})$. Fix $v \in C_c^\infty(\mathcal{V})$ and note that

$$(D^l u^\delta)(v) = \langle u^\delta, D^l v \rangle = \langle u^\delta - u, D^l v \rangle + \langle u, D^l v \rangle.$$

By assumption, we have that $|\langle u, D^l v \rangle| \lesssim \|v\|$, and so we are reduced to showing that $|\langle u^\delta - u, D^l v \rangle| \lesssim \|v\|$. Since D is local, D^l is local, and we have that $\text{spt } D^l v \subset \text{spt } v$. Choose $M \in \mathbb{N}$ such that $\sum_{i=1}^M \eta_i = 1$ on $\text{spt } v$, we then have

$$\begin{aligned} |\langle u^\delta - u, D^l v \rangle| &= \left| \left\langle \sum_{i=1}^N u_i^{\delta_i} - \sum_{i=1}^N u_i, D^l v \right\rangle \right| \leq \sum_{i=1}^N |\langle u_i^{\delta_i} - u_i, D^l v \rangle| \\ &\leq \sum_{i=1}^N \frac{\delta}{2^i} \|v\| \leq \delta \|v\|. \end{aligned}$$

Thus, $u^\delta \in \mathcal{S}_l(D)$. By a similar argument, we obtain that $u^\delta \rightarrow u$ in the graph norm $\|\cdot\|_{D^l} = \|D^l \cdot\| + \|\cdot\|$. \square

With the aid of this proposition, we prove Theorem 2.4.

Proof of Theorem 2.4. If $\mathcal{D}_0^l(D) = \mathcal{D}^l(D)$, then $(D^l)_D = (D^l)_N = (D_c^l)^*$, where the last equality follows from Proposition 4.2. That is, $(D^l)_D = (D^l)_D^*$, which shows that $(D^l)_D$ is self-adjoint.

Now, if $(D^l)_D$ is self-adjoint, we have that $(D^l)_D = (D^l)_D^* = (D_N^l)$ and so $(D^l)_N$ is self-adjoint. If $(D^l)_N$ is self-adjoint, since $\mathcal{S}_l(D) \subset \mathcal{D}(D^l)_N$, it follows that $(l\text{-Neg})$ is satisfied.

Recall the notation $D_2^l = D_l$ with domain $\mathcal{D}(D_2^l) = \mathcal{S}_l(D)$. Assuming the negligible boundary condition, we obtain that $(D^l)_N \subset (D_2^l)^*$. But since $D_c^l \subset D_2^l$, we have that $(D_2^l)^* \subset (D_c^l)^* = (D_N^l)$ by Proposition 4.2. So, $(D_2^l)^* = (D_c^l)^*$ and so by considering the bi-adjoint, we obtain that $(D^l)_N = (D^l)_D$.

The essential self-adjointness of D_c^l is equivalent to the self-adjointness of $(D^l)_D$. \square

5. DENSITY PROPERTIES FROM HIGHER TO LOWER POWERS

In this section, we prove that, if $(D^m)_0 = \overline{D_c^m}$ is self-adjoint, then so is every power $(D^l)_0$ for $l = 1, \dots, m$. The reasons for this involve no geometry, but only properties of the operators, and to highlight that, we will keep the presentation sufficiently abstract. The way we will proceed is to move from the operator $(D^m)_0$ to the operator $|(D^m)_0|$, then use results from interpolation theory to assert that $|(D^m)_0|^\alpha = (\sqrt{(D^m)_0^2})^\alpha$ has $C_c^\infty(\mathcal{V})$ as a core. We then employ an alternative argument to show that this is a core for D_0 , then use functional calculus and operator theory to assert that $|(D^m)_0|^\alpha = |D_0|^{m\alpha}$.

The functional calculus we use here is the holomorphic functional calculus. For self-adjoint operators T , this functional calculus is given by:

$$\psi(T)u = \frac{1}{2\pi} \oint_\gamma \psi(\zeta)(\zeta I - T)^{-1}u \, d\zeta,$$

where γ is a curve cutting the spectrum at zero and infinity inside a bisector. The functions ψ are holomorphic on a bisector and decay sufficiently rapidly at 0 and at ∞ . The functional calculus can be extended to bounded holomorphic functions f on a bisector. A detailed exposition of these ideas can be found in [1] by Albrecht, Duong, McIntosh and in the book [13] by Haase.

Lemma 5.1. *Let T be a non-negative self-adjoint operator on a Hilbert space \mathcal{H} , and let $\mathcal{C} \subset \mathcal{D}(T)$ be a core for T . Then, \mathcal{C} is a core for T^α for all $\alpha \in [0, 1]$.*

Proof. The case $\alpha = 0, 1$ are easy, so we fix $\alpha \in (0, 1)$.

First we show that $\mathcal{D}(T)$ is dense in $\mathcal{D}(T^\alpha)$. Theorem 6.6.1 in [13] yields that the real-interpolation space, $(\mathcal{H}, \mathcal{D}(T))_{\alpha, p}$ has $\mathcal{D}(T)$ as a dense subspace for $\alpha \in (0, 1)$ and $p \in [1, \infty)$. By Theorem 4.3.12 in [15], on choosing $p = 2$, we obtain that $(\mathcal{H}, \mathcal{D}(T))_{\alpha, 2} = \mathcal{D}(T^\alpha)$.

Next, note that $\mathcal{D}(T) \subset \mathcal{D}(T^\alpha)$ by functional calculus since $T^\alpha = (I+T)f_\alpha(T)$, where $f_\alpha(\zeta) = \zeta^\alpha/(1+\zeta)$. Moreover, since $f_\alpha(T) \in \mathcal{L}(\mathcal{H})$, we get that $\|T^\alpha u\| \lesssim \|(I+T)u\|$ for all $u \in \mathcal{D}(T)$.

Fix $u \in \mathcal{D}(T^\alpha)$, since we have already proved that $\mathcal{D}(T)$ is dense in $\mathcal{D}(T^\alpha)$, we can find $v_n \in \mathcal{D}(T)$ such that

$$\|u - v_n\| \leq \frac{1}{2^n} \quad \text{and} \quad \|T^\alpha u - T^\alpha v_n\| \leq \frac{1}{2^n}.$$

Moreover, since \mathcal{C} is a core for T , we can find $u_n \in \mathcal{C}$ such that

$$\|u_n - v_n\| < \frac{1}{2^n} \quad \text{and} \quad \|Tu_n - Tv_n\| < \frac{1}{2^n}.$$

Thus, on combining this with the estimate $\|T^\alpha u\| \lesssim \|(I+T)u\|$ for $u \in \mathcal{D}(T)$, and since $v_n \in \mathcal{D}(T)$,

$$\begin{aligned} \|T^\alpha u - T^\alpha u_n\| &\leq \|T^\alpha u_n - T^\alpha v_n\| + \|T^\alpha v_n - T^\alpha u\| \\ &\lesssim \|(I+T)(u_n - v_n)\| + \frac{1}{2^n} \leq \|u_n - v_n\| + \|Tu_n - Tv_n\| + \frac{1}{2^n} < \frac{3}{2^n}. \end{aligned}$$

This shows that \mathcal{C} is a core for T^α . \square

In the following lemma, we note that for a self-adjoint operator T , $|T| = \sqrt{T^2}$.

Lemma 5.2. *Let T be self-adjoint on \mathcal{H} . Then, for any integer $k \geq 0$, $\mathcal{D}(|T|^k) = \mathcal{D}(T^k)$ with $\||T|^k u\| = \|T^k u\|$.*

Proof. Consider the functions

$$f_1(\zeta) = \frac{|\zeta|^k}{1 + \zeta^k}, \quad \text{and} \quad f_2(\zeta) = \frac{\zeta^k}{1 + |\zeta|^k}.$$

Each such function is holomorphic, bounded, and hence, $f_1(T) = |T|^k (I + T^k)^{-1} \in \mathcal{L}(\mathcal{H})$ and $f_2(T) = T^k (I + |T|^k)^{-1} \in \mathcal{L}(\mathcal{H})$. This shows that $\mathcal{D}(T^k) = \mathcal{D}(|T|^k)$ in graph norm.

To obtain the equivalence of norms, we note that by the self-adjointness of T , $\|Tu\| = \||T|u\|$. For higher powers,

$$\|T^k u\| = \||T| T^{k-1} u\| = \|T^{k-1} |T| u\| = \cdots = \||T|^k u\|,$$

where the second equality follows from functional calculus. \square

For the next lemma, we specialise to the operator in question.

Lemma 5.3. *Let D be an operator satisfying (A1)-(A2). If $(D^l)_D$ is self-adjoint, then D_D is also self-adjoint.*

Proof. First, we consider the case that $l = 2$. The symmetry condition (A1) implies that D_D exists and $D \subset D_D$ so that $D^2 \subset (D_D)^2$. Since $(D_D)^2$ is closed, $(D^2)_D \subset (D_D)^2$ and hence, by Lemma 3 in [5] (with $T = D_D$ and $S = (D_D)^2$), D_D is self-adjoint. Thus $D_D^2 = (D_D)^2$.

Now, for $l = 2^m$, we replace D by $D^{2^{m-1}}$ and by this argument, we obtain that $(D^{2^{m-1}})_D$ is self-adjoint whenever $(D^{2^m})_D$ is self-adjoint. Repeating this procedure m -times, we obtain that D_D is self-adjoint.

Next, suppose that l is an odd number. Assume for contradiction that D_D is not self-adjoint.

Then $\mathcal{N}(D_D^* + \iota) \neq 0$ or $\mathcal{N}(D_D^* - \iota) \neq 0$. Without loss of generality, we assume $\mathcal{N}(D_D^* - \iota) \neq 0$ and we can find a non-zero vector $v \in \mathcal{D}(D_D^*)$ such that $D_D^* v = D^* v = \iota v$.

By Proposition 4.2, we have that $\mathcal{D}(D^*) = \mathcal{D}(D_N) = \mathcal{D}_\infty(D)$, so $D_N v = \iota v$ which implies that $v \in \mathcal{D}(D_N^a)$ for all $a \geq 1$. Invoking (A1), since $v \in \mathcal{D}(D_N^a)$ for all $a \leq l$,

$$\langle D_N^l v, u \rangle = \langle D_N^{l-1} v, D_c u \rangle = \cdots = \langle v, D_c^l u \rangle$$

for all $u \in C_c^\infty(\mathcal{V})$. That is, $v \in \mathcal{D}(D_c^{l*}) = \mathcal{D}_\infty^l(D)$. But $(D^l)_D$ is self-adjoint, $\mathcal{D}_\infty^l(D) = \mathcal{D}((D^l)_D)$, and so we have $(D^l)_D = D_N^l v = \iota v$, which contradicts that $(D^l)_D$ is self-adjoint.

For a general l , write $l = a \cdot 2^b$ for a odd, we obtain that D_D is self-adjoint whenever $(D^l)_D$ is self-adjoint. \square

With the aid of this, we obtain a proof of Theorem 2.5.

Proof of Theorem 2.5. First, $D \subset D_D$ yields that $D^l \subset D_D^l$ and so $(D^l)_D \subset D_D^l$. The operator $(D^l)_D$ is self-adjoint by hypothesis and D_D^l is self-adjoint since D_D is self-adjoint. Then, by Lemma 5.3, we have that $(D^l)_D = D_D^l$.

By Lemma 5.1, $C_c^\infty(\mathcal{V})$ is a core for $|(D^l)_D|^\alpha = |D_D^l|^\alpha$. On considering the functions

$$f_1(\zeta) = \frac{1}{1 + (|\zeta|^l)^\alpha} \quad \text{and} \quad f_2(\zeta) = \frac{1}{1 + |\zeta|^{l\alpha}},$$

and observing that $f_1 = f_2$ is bounded holomorphic on \mathbb{C} , we obtain

$$(1 + (|D_D|^l)^\alpha)^{-1} = f_1(D) = f_2(D) = (1 + (|D_D|^{l\alpha})^{-1},$$

and hence, $(|D_D|^l)^\alpha = |D_D|^{l\alpha}$. Thus, we have that $C_c^\infty(\mathcal{V})$ is a core for $|D_D|^{l\alpha}$ for $\alpha \in [0, 1]$.

On setting $\alpha = k/l$ for integers $k = 1, \dots, l$, by Lemma 5.2, we obtain that $C_c^\infty(\mathcal{V})$ is a core for D_D^k . \square

6. ESSENTIAL SELF-ADJOINTNESS OF POWERS IN THE COMPLETE SETTING

Throughout this section, we assume that g is a rough metric that induces a *complete length* space. We will denote the distance metric of this length space by d . Furthermore, we assume the bundle \mathcal{V} is equipped with a rough metric h .

As a background assumption, we assume (A1)-(A2) for the operator D (i.e., for the case $l = 1$).

Proposition 6.1. *The formula $\langle D_N u, v \rangle = \langle u, D v \rangle$ holds whenever $u \in \mathcal{D}(D_N)$ and $v \in C_c^{0,1}(\mathcal{M})$.*

Proof. We note by (A1) that $\langle D u, v \rangle = \langle u, D_D v \rangle$ for all $u \in \mathcal{S}_1$ and $v \in \mathcal{D}(D_D)$. Thus, it suffices to show that $C_c^{0,1}(\mathcal{V}) \subset \mathcal{D}(D_D)$. For that, fix $v \in C_c^{0,1}(\mathcal{V})$. On fixing a smooth partition of unity subordinate to locally comparable precompact charts, we can further assume that $\text{spt } v \subset U$, where U is a precompact open chart. Inside there, fix the flat connection ∇^U , and hence, by (A2), we can deduce that that whenever $D u \in L^2(\mathcal{V})$ with $\text{spt } u \subset U$, implies that $u \in W^{1,2}(U, \mathcal{V})$ with the estimate $\|D u\|_U \lesssim \|\nabla^U u\|_U + \|u\|_U$. Now, let $v_n = (J_n^{\frac{1}{n}} * v^i) e_i$, where $\{e_i\}$ is the frame in U for which ∇^F is flat and $J_n^{\frac{1}{n}}$ is the standard symmetric mollifier J^ε with $\varepsilon = 1/n$. Hence, we have that $\text{spt } v_n \subset U$ for n large enough, and moreover, $v_n \in C_c^\infty(\mathcal{V})$. In particular, $D v_n \in L^2(\mathcal{V})$ and moreover, $v_n \rightarrow v$ in $L^2(\mathcal{V})$. Thus,

$$\|D_D v_n - D_D v_m\| \lesssim \|\nabla^U v_n - \nabla^U v_m\|_U + \|v_n - v_m\|$$

and the right hand side tends to zero as $m, n \rightarrow \infty$. By the closedness of D_D , we have that $v \in \mathcal{D}(D_D)$. It is easy to see that this now extends for $u \in \mathcal{D}(D_N)$. \square

When we apply this proposition in later parts, we will be taking $u \in \mathcal{S}_k$ and considering the section ηu , where $\eta \in C_c^\infty(\mathcal{M})$. In this case we have that $D^k(\eta u)$ is $C_c^{m-k}(\mathcal{V})$ and hence this lies in $\mathcal{D}(D_D) \subset \mathcal{D}(D_N)$. Thus, we can apply this proposition to this case.

6.1. Some preliminary constructions and remarks. In [9], Chernoff defines the *local velocity of propagation* by:

$$c(x) = \sup\{|M_f(x)|_{\text{op}} : f \in C^\infty(\mathcal{M}), |\nabla(f)(x)|_g = 1\},$$

and the *velocity of propagation inside a ball* by $c(r) = \text{esssup}\{c(x) : x \in B_r\}$. It is easy to see that $c(r)$ is non-negative and monotonically increasing.

Since $D = A^i \partial_i + B$ locally with $A_i \neq 0$ for some i , we have that $c(r) \neq 0$ for all $r > 0$. Also, since the operator locally takes the form $A^i \partial_i + B$, it is easy to see that $M_f(x)u(x) = (\partial_i f)(x)A^i u(x)$. By covering the ball B_r by a finite number of locally comparable precompact charts, and using that $A^i \in L^\infty_{\text{loc}}$, we obtain: there exists $C_{B_r} < \infty$ such that $|M_f(x)| \leq C_{B_r} |\nabla f(x)|$ for almost-every $x \in B_r$. Therefore, for each $r > 0$, the velocity of propagation satisfies $c(r) < \infty$.

If we take a smooth function f , we then find that for any smooth section u

$$|M_f(u)(x)|_h \leq |\nabla(f)(x)|_g c(x) |u(x)|_h \quad (6.1)$$

for almost-every $x \in \mathcal{M}$. Letting $\|\cdot\|_{B_r}$ denote the L^2 norm on the ball B_r , we obtain

$$\|M_f(u)\|_{B_r} \leq (\text{esssup}\{|\nabla(f)(x)|_g : x \in B_r\}) c(r) \|u\|_{B_r}. \quad (6.2)$$

The following construction is based on Wolf's construction in §5 of [18], and adapted to the case of rough metrics.

Fix a point x_0 in \mathcal{M} , and for any point $x \in \mathcal{M}$, let $\rho(x) := d(x, x_0)$, where d denotes the distance function associated to the length structure induced on \mathcal{M} by g . The triangle inequality shows us that $|\rho(y) - \rho(x)| \leq d(y, x)$ for any $x, y \in \mathcal{M}$. Thus ρ is Lipschitz and hence differentiable almost-everywhere. Inside a locally comparable chart (U, ψ) , we have $C_U \geq 1$ such that $|\nabla \rho(x)|_{g(x)} \leq C_U$ for almost-every $x \in U$.

For $r > 0$ we let B_r denote the open ball centred about the fixed point x_0 . By the assumption that our rough metric induces a complete length space, we are able to apply the metric space version of the Hopf-Rinow theorem (see Proposition 3.7 in [6]) and obtain that metric balls B_r are precompact. As a consequence, on a ball B_r , there exists $C_{B_r} \in [1, \infty)$ such that $|\nabla \rho(x)| \leq C_{B_r}$ for almost-every $x \in B_r$. Define

$$\tilde{d}(r) = \max(1, \text{esssup}\{|\nabla \rho(x)| : x \in B_r\}),$$

and note that $\tilde{d}(r)$ is increasing in r . Also, define $e(r) = \tilde{d}(r)c(r)$ which is again an increasing function in r .

Choose a smooth function $a : \mathbb{R} \rightarrow [0, 1]$ such that $a(-\infty, 1] = 1$, $a[2, \infty) = 0$, and such that a is non-zero on the interval $(1, 2)$. Denote $M = \max|a'(t)|$. For each $r > 0$, define $b_r : \mathcal{M} \rightarrow [0, 1]$ by $b_r(y) = a(\frac{\rho(y)}{e(r)r})$. Then $b_r = 1$ on $B_{e(r)r}$ and $\text{spt}(b_r) \subseteq \overline{B_{2e(r)r}}$. Furthermore b_r is non-zero on $B_{2e(r)r}$, and since $B_{2e(r)r}$ is precompact, it follows that b_r is compactly supported. The Lipschitz property of ρ implies that b_r is Lipschitz and hence almost-everywhere differentiable. At points of

differentiability, and points where $|\cdot|_g$ is defined, we have

$$\left| \nabla(b_r) \right|_g^2 = \left| \frac{1}{e(r)r} a' \left(\frac{\rho}{e(r)r} \right) \nabla(\rho) \right|_g^2 \leq \frac{M^2}{c(r)^2 r^2}. \quad (6.3)$$

We remind the reader that $|\cdot|_g$ is defined almost-everywhere so the above gradient bound holds almost-everywhere on \mathcal{M} . In particular, $\text{esssup} \{ |\nabla(b_r)(x)| : x \in \mathcal{M} \}$ is bounded above by $M(c(r)r)^{-1}$.

Using this function b_r in equation (6.2) we find that

$$\|M_{b_r}(u)\|_{B_{2e(r)r}} \leq \frac{M}{c(r)r} c(r) \|u\|_{B_{2e(r)r}} = \frac{M}{r} \|u\|_{B_{2e(r)r}}. \quad (6.4)$$

We will also be making use of the powers b_r^k for $k \geq 1$. In this case we note that because $\nabla(b_r^k) = k b_r^{k-1} \nabla(b_r)$, we get the following estimate:

$$\|M_{b_r^k}(u)\|_{B_{2e(r)r}} \leq \frac{kM}{r} \|b_r^{k-1} u\|_{B_{2e(r)r}}. \quad (6.5)$$

6.2. Essential self-adjointness of D . In this subsection we prove that D is essentially self-adjoint on $C_c^\infty(\mathcal{V})$. The approach we will be taking is to establish that D satisfies the negligible boundary condition.

Theorem 6.2. *Let \mathcal{V} be a vector bundle with a rough metric h over a manifold \mathcal{M} with a rough metric g inducing a complete length space. If the operator D satisfies (A1)-(A2), then it satisfies the negligible boundary condition (l-Neg), and D on $C_c^\infty(\mathcal{V})$ is essentially self-adjoint.*

Proof. We fix two arbitrary sections $u, v \in \mathcal{S}_1$, what we need to prove is that

$$\langle Du, v \rangle = \langle u, Dv \rangle.$$

In order to do this, we consider the truncated section $b_r v$ for some $r > 0$. This section is Lipschitz with compact support. We then have that

$$\langle Du, b_r v \rangle = \langle u, D(b_r v) \rangle = \langle u, M_{b_r} v \rangle + \langle u, b_r Dv \rangle$$

where the first equality comes from Proposition 6.1, and the second from the fact that $M_{b_r} = [D, b_r I]$.

From (6.1) and (6.4) we see that $M_{b_r} v \rightarrow 0$ as $r \rightarrow \infty$. Furthermore, $b_r u \rightarrow u$ in L^2 as $r \rightarrow \infty$ by the dominated convergence theorem, and similarly for $b_r Dv$. This implies that if we let $r \rightarrow \infty$ in the above equality, we obtain

$$\langle Du, v \rangle = \langle u, Dv \rangle.$$

Since u, v were arbitrary sections it follows that such an equality holds for all $u, v \in \mathcal{S}_1$, which is the negligible boundary condition (l-Neg). The conclusion follows from Theorem 2.4. \square

6.3. Essential self-adjointness of D^k . In this subsection we will show that the powers D^k on $C_c^\infty(\mathcal{V})$, for $1 \leq k \leq m+1$, are essentially self-adjoint when the operator D has C^m coefficients. The case $k = 1$ holds even for operators with L_{loc}^∞ coefficients and was obtained in Theorem 6.2. We do not expect the essential self-adjointness to hold in general for D^k with $k > m+1$ since even for smooth, compactly supported sections u , we can only make sense of $D^k u$ distributionally.

Our approach will be based on certain local estimates over a ball of radius r . We will then show that these local estimates prolong to global estimates on the whole manifold by taking the radius r to infinity. These global estimates allow us to prove the negligible boundary condition, and from our previous work, establish essential self-adjointness. This idea of using local estimates to establish essential self-adjointness of an operator is inspired by Wolf's work on the Dirac operator in [18]. More specifically, in Proposition 6.2 in [18], Wolf establishes a certain global estimate of $\|D(u)\|$ in terms of $\|D^2(u)\|$ and $\|u\|$ and this, along with some other facts, allows him to conclude the essential self-adjointness of D^2 , where D is the Dirac operator. We will show that when $u \in \mathcal{S}_{k+1}$ one can bound $\|D^k(u)\|$ in terms of $\|D^{k+1}(u)\|$ and $\|u\|$. This implies that $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$, which we will then exploit to prove the negligible boundary condition (*l-Neg*) for D^{k+1} . We should also mention that Proposition 6.2 in [18] is based on the method of Andreotti-Vesentini (see §6 in [2]).

In addition to the assumptions of Theorem 6.2, for the remainder of this section, we assume that D is a C^m coefficient operator satisfying (A1)-(A2) for D^k with $2 \leq k \leq m+1$.

Proposition 6.3. *For a section $u \in \mathcal{S}_2$, we have that for any $t_1, t_2 > 0$ that*

$$\|D(u)\|^2 \leq \frac{1}{2t_2} \|D^2(u)\|^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \|u\|^2. \quad (6.6)$$

In particular, $D(u) \in L^2(\mathcal{V})$ which implies $u \in \mathcal{S}_1$.

Proof. We will be using the functions b_r as constructed in §6.1. We start by proving the following:

If $t_1, t_2 > 0$ are given then there exists $r_1 = r_1(t_1)$ such that for all $r \geq r_1$

$$\left(1 - \frac{2M^2}{t_1 r^2}\right) \|b_r D(u)\|_{B_{2e(r)r}}^2 \leq \frac{1}{2t_2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \|u\|_{B_{2e(r)r}}^2 \quad (6.7)$$

and $\left(1 - \frac{2M^2}{t_1 r^2}\right) > 0$.

Fix a smooth section u of \mathcal{V} . Consider the compactly supported section $b_r^2 u$. Since b_r is differentiable almost-everywhere, at points of differentiability, we have the following formula:

$$D(b_r^2 u) = M_{b_r^2}(u) + b_r^2 D(u).$$

We then fix a smooth compactly supported function η such that $\eta = 1$ on $B_{3e(r)r}$ and $\eta = 0$ outside of $B_{4e(r)r}$. We have

$$\|b_r D(\eta u)\|^2 = \langle b_r D(\eta u), b_r D(\eta u) \rangle = \langle b_r^2 D(\eta u), D(\eta u) \rangle = \langle D(b_r^2 D(\eta u)), \eta u \rangle$$

where we have used Proposition 6.1 to get the last equality, noting that $b_r^2 D(\eta u) \in C_c^{0,1}(M)$. As b_r^2 has support contained in $\overline{B_{2e(r)r}}$, we have that the above inner products are zero on the complement of $\overline{B_{2e(r)r}}$. This implies, using the fact that $\eta = 1$ on $B_{3e(r)r}$, that

$$\|b_r D(u)\|_{B_{2e(r)r+\varepsilon}}^2 = \langle D(b_r^2 D(u)), u \rangle_{B_{2e(r)r+\varepsilon}}$$

for every $\varepsilon > 0$ sufficiently small. Therefore it must be true on the open ball $B_{2e(r)r}$, so we find

$$\begin{aligned} \|b_r D(u)\|_{B_{2e(r)r}}^2 &= \langle D(b_r^2 D(u)), u \rangle_{B_{2e(r)r}} \\ &= \langle M_{b_r^2}(Du), u \rangle_{B_{2e(r)r}} + \langle b_r^2 D^2(u), u \rangle_{B_{2e(r)r}}. \end{aligned} \quad (6.8)$$

Using Cauchy-Schwarz and the estimate (6.5), for any $t_1 > 0$ we obtain the bound

$$|\langle M_{b_r^2}(Du), u \rangle_{B_{2e(r)r}}| \leq \frac{2M^2}{t_1 r^2} \|b_r D(u)\|_{B_{2e(r)r}}^2 + \frac{t_1}{2} \|u\|_{B_{2e(r)r}}^2. \quad (6.9)$$

For any $t_2 > 0$, the Cauchy-Schwarz inequality also implies

$$|\langle b_r^2 D^2(u), u \rangle_{B_{2r}}| \leq \frac{1}{2t_2} \|b_r^2 D^2(u)\|_{B_{2r}}^2 + \frac{t_2}{2} \|u\|_{B_{2r}}^2. \quad (6.10)$$

Using (6.9) and (6.10) in (6.8) we obtain

$$\|b_r D(u)\|_{B_{2e(r)r}}^2 \leq \frac{2M^2}{t_1 r^2} \|b_r D(u)\|_{B_{2e(r)r}}^2 + \frac{1}{2t_2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \|u\|_{B_{2e(r)r}}^2,$$

which we can simplify to

$$\left(1 - \frac{2M^2}{t_1 r^2}\right) \|b_r D(u)\|_{B_{2e(r)r}}^2 \leq \frac{1}{2t_2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \|u\|_{B_{2e(r)r}}^2.$$

By choosing r large enough we can find a $r_1 = r_1(t_1)$ such that $\left(1 - \frac{2M^2}{t_1 r^2}\right) > 0$ for any $r \geq r_1(t_1)$, and hence we have proved (6.7).

Using the fact that $b_r \leq 1$ and $b_r = 1$ on B_r we obtain

$$\left(1 - \frac{2M^2}{t_1 r^2}\right) \|D(u)\|_{B_r}^2 \leq \frac{1}{2t_2} \|D^2(u)\|_{B_{2r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \|u\|_{B_{2r}}^2$$

for any $r \geq r_1$. Letting $r \rightarrow \infty$ then establishes (6.6) by the monotone convergence theorem. \square

Proposition 6.3 immediately allows us to prove that D^2 must be essentially self-adjoint.

Proposition 6.4. D^2 is essentially self-adjoint on \mathcal{M} .

Proof. The above Proposition 6.3 implies that $\mathcal{S}_2 \subseteq \mathcal{S}_1$. Furthermore, Theorem 6.2 implies that D satisfies the negligible boundary condition (l -Neg) on \mathcal{M} . Therefore, if we take $u, v \in \mathcal{S}_2$ we have

$$\langle D^2 u, v \rangle = \langle Du, Dv \rangle = \langle u, D^2 v \rangle,$$

which is the negligible boundary condition (l -Neg) for D^2 . From Theorem 2.4 it follows that D^2 is essentially self-adjoint on \mathcal{M} . \square

Remark 6.5. We observe that part of the hypothesis of Proposition 6.3 involved assuming that our section $u \in \mathcal{S}_2$. This was done so that in the global estimate (6.6), we knew that the right hand side was finite, hence this gives us that the left hand side is finite. In the case of the local estimate (6.7), the assumption that $u \in \mathcal{S}_2$ is not needed. It suffices to assume that u is smooth (in fact C^2 is enough). This is because we can fix a smooth compactly supported function η taking the value $\eta = 1$ on $B_{3e(r)r}$ and which vanishes outside of $B_{4e(r)r}$. Then, $\eta u \in \mathcal{S}_2$ and we can use the fact that $\eta u = u$ on $B_{2e(r)r}$ where the local estimate holds. We will see that this is a crucial observation in the argument to obtain local estimates for higher $D^k u$.

In Proposition 6.3, in order to obtain our required global estimate (6.6), we first proved an estimate on the ball $B_{2e(r)r}$ before taking a limit $r \rightarrow \infty$. In order to prove the essential self-adjointness of higher powers D^{k+1} for $k \geq 2$, we will proceed along the same lines. Our goal will therefore be to obtain a similar bound on $\|D^k(u)\|^2$ in terms of $\|D^{k+1}(u)\|^2$ and $\|u\|^2$, over balls of sufficiently large radius. We will give the details for the case $k = 2$, which involves some slight modifications of the above $k = 1$ case, and then we will show how to do the general case via induction.

We start by proving the following local estimate.

Proposition 6.6. For $u \in \mathcal{S}_3$, given any $t_1, t_3, t_4 > 0$ we can choose r and t_2 sufficiently large so that

$$\begin{aligned} (i) \quad & C_1(r, t_1) = \left(1 - \frac{2M^2}{t_1 r^2}\right) > 0, \text{ and} \\ (ii) \quad & C_2(r, t_1, t_2, t_3, t_4) = \left(1 - \frac{8M^2}{t_3 r^2} - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) C_1(r, t_1)^{-1} \frac{1}{2t_2}\right) > 0. \end{aligned}$$

Moreover, the following estimate holds:

$$\begin{aligned} C_2 \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 &\leq \frac{1}{2t_4} \|b_r^3 D^3(u)\|_{B_{2e(r)r}}^2 \\ &\quad + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \left(\frac{t_3}{2} + \frac{t_4}{2}\right) C_1^{-1} \|u\|_{B_{2e(r)r}}^2. \end{aligned} \tag{6.11}$$

Proof. We start by estimating $\|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2$. As in the proof of Proposition 6.3 we fix a smooth compactly supported function η such that $\eta = 1$ on $B_{3e(r)r}$ and $\eta = 0$ outside of $B_{4e(r)r}$. We then have

$$\begin{aligned} \langle b_r^2 D^2(\eta u), b_r^2 D^2(\eta u) \rangle &= \langle D(b_r^4 D^2(\eta u)), D(\eta u) \rangle \\ &= \langle M_{b_r^4}(D^2(\eta u)), D(\eta u) \rangle + \langle b_r^4 D^3(\eta u), D(\eta u) \rangle \\ &= \langle M_{b_r^4}(D^2(\eta u)), D(\eta u) \rangle + \langle b_r^3 D^3(\eta u), b_r D(\eta u) \rangle \end{aligned}$$

where we have used Proposition 6.1 to obtain the first equality (noting that $D(\eta u) \in C_c^{m-1}(E)$, hence is in $\mathcal{D}(D_D)$), and the fact that at points of differentiability of b_r^4 , $D(b_r^4 D^2(\eta u)) = M_{b_r^4}(D^2(\eta u)) + b_r^4 D^3(\eta u)$. Since b_r has support contained $\overline{B_{2e(r)r}}$ and $\eta = 1$ on $B_{3e(r)r}$, the above gives

$$\langle b_r^2 D^2(u), b_r^2 D^2(u) \rangle_{B_{2e(r)r+\varepsilon}} = \langle M_{b_r^4}(D^2(u)), D(u) \rangle + \langle b_r^3 D^3(u), b_r D(u) \rangle_{B_{2e(r)r+\varepsilon}}$$

for any $\varepsilon > 0$ sufficiently small. By taking $\varepsilon \rightarrow 0$ we obtain

$$\langle b_r^2 D^2(u), b_r^2 D^2(u) \rangle_{B_{2e(r)r}} = \langle M_{b_r^4}(D^2(u)), D(u) \rangle_{B_{2e(r)r}} + \langle b_r^3 D^3(u), b_r D(u) \rangle_{B_{2e(r)r}}.$$

Applying Cauchy-Schwarz, the estimate (6.5), and the fact that b_r has no zeroes in $B_{2e(r)r}$, we obtain

$$\begin{aligned} |\langle M_{b_r^4}(D^2(u)), D(u) \rangle_{B_{2e(r)r}}| &= \left| \left\langle \frac{1}{b_r} M_{b_r^4}(D^2(u)), b_r D(u) \right\rangle_{B_{2e(r)r}} \right| \\ &\leq \frac{8M^2}{t_3 r^2} \left\| \frac{b_r^3}{b_r} D^2(u) \right\|_{B_{2e(r)r}}^2 + \frac{t_3}{2} \|b_r D(u)\|_{B_{2e(r)r}}^2 \\ &= \frac{8M^2}{t_3 r^2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 + \frac{t_3}{2} \|b_r D(u)\|_{B_{2e(r)r}}^2. \end{aligned}$$

Similarly we get the bound

$$|\langle b_r^3 D^3(u), b_r D(u) \rangle_{B_{2e(r)r}}| \leq \frac{1}{2t_4} \|b_r^3 D^3(u)\|_{B_{2e(r)r}}^2 + \frac{t_4}{2} \|b_r D(u)\|_{B_{2e(r)r}}^2.$$

Using these two estimates we get

$$\begin{aligned} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 &\leq \frac{8M^2}{t_3 r^2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_3}{2} + \frac{t_4}{2} \right) \|b_r D(u)\|_{B_{2e(r)r}}^2 \\ &\quad + \frac{1}{2t_4} \|b_r^3 D^3(u)\|_{B_{2e(r)r}}^2. \end{aligned}$$

We now want to plug in our local estimate for the $\|b_r D(u)\|_{B_{2e(r)r}}^2$ term occurring on the right. We remind the reader that at first sight this seems impossible. In obtaining a local estimate for $D(u)$ we assumed $u \in \mathcal{S}_2$, and our hypothesis at this point is that $u \in \mathcal{S}_3$. However, we do not know, a priori, that $u \in \mathcal{S}_2$ and in fact, this is what we will prove when we obtain the associated *global* estimate. However, as we observed in Remark 6.5, we actually do not require u to be in \mathcal{S}_2 for the *local* estimate. It suffices to ask that u be smooth. Thus, on substituting the local estimate (6.7) into the above inequality, we obtain:

$$\begin{aligned} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 &\leq \frac{8M^2}{t_3 r^2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 \\ &\quad + \left(\frac{t_3}{2} + \frac{t_4}{2} \right) \left(1 - \frac{2M^2}{t_1 r^2} \right)^{-1} \frac{1}{2t_2} \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 \\ &\quad + \left(\frac{t_1}{2} + \frac{t_2}{2} \right) \left(\frac{t_3}{2} + \frac{t_4}{2} \right) \left(1 - \frac{2M^2}{t_1 r^2} \right)^{-1} \|u\|_{B_{2e(r)r}}^2 \\ &\quad + \frac{1}{2t_4} \|b_r^3 D^3(u)\|_{B_{2e(r)r}}^2. \end{aligned}$$

We can re-write this to give

$$\begin{aligned} &\left(1 - \frac{8M^2}{t_3 r^2} - \left(\frac{t_3}{2} + \frac{t_4}{2} \right) C_1^{-1} \frac{1}{2t_2} \right) \|b_r^2 D^2(u)\|_{B_{2e(r)r}}^2 \\ &\leq \frac{1}{2t_4} \|b_r^3 D^3(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2} \right) \left(\frac{t_3}{2} + \frac{t_4}{2} \right) C_1^{-1} \|u\|_{B_{2e(r)r}}^2, \end{aligned}$$

where we remind the reader that $C_1(r, t_1) = \left(1 - \frac{2M^2}{t_1 r^2}\right)$.

We are now left with showing that we can choose r and t_2 sufficiently large so that the coefficients in the estimate are positive. It is easy to see that for large r we have $C_1 > 0$. We take $t_2 = \lambda_2(t_3 + t_4)$, where λ_2 is a parameter, and observe that by taking r and λ_2 sufficiently large, we can make the coefficient C_2 on the left hand side positive. This finishes the proof. \square

Observe that if we take $r \rightarrow \infty$, we have that $C_1 \rightarrow 1$, and $C_2 \rightarrow 1 - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \frac{1}{2t_2}$. So by choosing t_2 appropriately, $1 - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \frac{1}{2t_2}$ can always be made greater than zero. Therefore by taking $r \rightarrow \infty$ in the above local estimate, we obtain the following global estimate.

Theorem 6.7. *For $u \in \mathcal{S}_3$, given any $t_1, t_3, t_4 > 0$ we can choose t_2 sufficiently large so that $1 - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \frac{1}{2t_2} > 0$, and*

$$\left(1 - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \frac{1}{2t_2}\right) \|D^2(u)\|^2 \leq \frac{1}{2t_4} \|D^3(u)\|^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \|u\|^2. \quad (6.12)$$

In particular, $\|D^2(u)\| < \infty$, and hence $\mathcal{S}_3 \subseteq \mathcal{S}_2$.

The above theorem can now be used to establish essential self-adjointness of $D^3(u)$.

Proposition 6.8. *D^3 is essentially self-adjoint.*

Proof. The above theorem shows us that $\mathcal{S}_3 \subseteq \mathcal{S}_2$, and we know from Proposition 6.4 that $\mathcal{S}_2 \subseteq \mathcal{S}_1$. Therefore, if we take $u, v \in \mathcal{S}_3$ then

$$\langle D^3 u, v \rangle = \langle Du, D^2 v \rangle = \langle u, D^3 v \rangle$$

which is the negligible boundary condition (*l-Neg*) for D^3 . It follows from Theorem 2.4 that D^3 is essentially self-adjoint. \square

The general case of proving essential self-adjointness of D^{k+1} proceeds along similar lines. In obtaining a local estimate for $D^k(u)$, we will meet constants of the form $C_k(r, t_1, t_2, \dots, t_{2k-1}, t_{2k})$ that will depend on a given set of positive numbers $t_1, t_3, \dots, t_{2k-1}, t_{2k}$, and constants $C_{k-1}(r, t_1, \dots, t_{2(k-1)})$ that arise from the local estimate for $D^{k-1}(u)$. The goal will be to show that we can pick these constants to be positive, by making a suitable choice for $t_2, t_4, \dots, t_{2(k-1)} > 0$. For example, in the case of obtaining a local estimate for $D^3(u)$ in terms of $D^4(u)$ and u , the constant $C_3(r, t_1, t_2, t_3, t_4, t_5, t_6)$ takes the form

$$C_3(r, t_1, t_2, t_3, t_4, t_5, t_6) = \left(1 - \frac{18M^2}{t_5 r^2} - \left(\frac{t_5}{2} + \frac{t_6}{2}\right) C_2^{-1} \frac{1}{2t_4}\right),$$

where we remind the reader that

$$C_2(r, t_1, t_2, t_3, t_4) = \left(1 - \frac{8M^2}{t_3 r^2} - \left(\frac{t_3}{2} + \frac{t_4}{2}\right) C_1(r, t_1)^{-1} \frac{1}{2t_2}\right),$$

and $C_1(r, t_1) = \left(1 - \frac{2M^2}{t_1 r^2}\right)$. The type of question we will be faced with is: given t_1, t_3, t_5, t_6 can we choose r, t_2, t_4 so that $C_i > 0$ for $1 \leq i \leq 3$? The idea would be to choose r and t_4 sufficiently large so that $C_3 > 0$. The problem is that as we vary t_4 the constant C_2 also changes, so there is a non triviality that we need to prove. The way in which we proceed is to generalise the argument demonstrating how C_2 can be made positive. We define $t_4 = \lambda_4(t_5 + t_6)$, where λ_4 is a parameter to be chosen later. We then define $t_2 = \lambda_2(t_3 + t_4) = \lambda_2(t_3 + \lambda_4(t_5 + t_6))$, and substitute these into the formulas for C_2 and C_3 to obtain

$$C_2 = \left(1 - \frac{8M^2}{t_3 r^2} - C_1^{-1} \frac{1}{4\lambda_2}\right), \quad C_3 = \left(1 - \frac{18M^2}{t_5 r^2} - C_2^{-1} \frac{1}{2\lambda_4}\right).$$

In the above formula for C_3 , we can see that the C_2^{-1} term does not depend on λ_4 . Therefore, we start by choosing r sufficiently large so that $C_1 > 0$. Then we choose λ_2 and r sufficiently large so that $C_2 > 0$, and finally choose λ_4 and r sufficiently large so that $C_3 > 0$.

The general case of proving that the constants that come out can always be made positive follows in a similar fashion. We will now prove a lemma that shows how to do this.

Lemma 6.9. *Let $C_1(r, t_1) = 1 - \frac{2M^2}{t_1 r^2}$, and for $i \geq 2$ recursively define the functions*

$$C_i(r, t_1, \dots, t_{2i-1}, t_{2i}) = 1 - \frac{2i^2 M^2}{t_{2i-1} r^2} - \left(\frac{t_{2i-1}}{2} + \frac{t_{2i}}{2}\right) C_{i-1}(r, t_1, \dots, t_{2(i-1)})^{-1} \frac{1}{2t_{2(i-1)}}.$$

Then the functions C_i for $i \geq 2$ satisfy the following two conditions.

- (i) $C_i(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2i-2}(t_{2i-1} + t_{2i}), t_{2i-1}, t_{2i})$ (where we insert $\lambda_{2j}(t_{2j+1} + t_{2j+2})$ in the $2j$ position $1 \leq j < i$) is independent of $t_4, t_6, \dots, t_{2i-2}, t_{2i}$, and
- (ii) $C_i(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2i-2}(t_{2i-1} + t_{2i}), t_{2i-1}, t_{2i}) > 0$ for $\lambda_2, \lambda_4, \dots, \lambda_{2i-2}, r$ sufficiently large.

In particular, given any positive $t_1, t_3, \dots, t_{2i-1}, t_{2i}$ we can make $C_i(r, t_1, \dots, t_{2i-1}, t_{2i}) > 0$ for $t_2, t_4, \dots, t_{2i-2}$ and r sufficiently large.

Proof. We will prove this by induction. We have already seen that it is true for the $i = 2$ case (and in fact the $i = 3$ case). So assume it is true for all $2 \leq i \leq k-1$, we will prove it is true for $i = k$.

The formula for C_k is given by

$$C_k(r, t_1, \dots, t_{2k-1}, t_{2k}) = 1 - \frac{2k^2 M^2}{t_{2k-1} r^2} - \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2}\right) C_{k-1}(r, t_1, \dots, t_{2(k-1)})^{-1} \frac{1}{2t_{2(k-1)}}.$$

We observe that if we insert $\lambda_{2j}(t_{2j+1} + t_{2j+2})$ into the $2j$ -th position for $1 \leq j < k-1$, we only affect the C_{k-1}^{-1} in the above formula for C_k . Our induction hypothesis tells us that this term can be made independent of $t_4, \dots, t_{2(k-1)}$. If we then substitute $\lambda_{2(k-1)}(t_{2k-1} + t_{2k})$ into the $2(k-1)$ -th position, we see that we do not affect the C_{k-1}^{-1} term, because this has been made independent of $t_{2(k-1)}$, and then we see

that we get rid of the t_{2k} dependence arising in the other terms. In particular, it follows that $C_k(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2k-2}(t_{2k-1} + t_{2k}), t_{2k-1}, t_{2k})$ is independent of $t_4, t_6, \dots, t_{2(k-1)}, t_{2k}$, and this establishes that C_k satisfies the first condition.

For the second condition, we observe that $C_k(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2k-2}(t_{2k-1} + t_{2k}), t_{2k-1}, t_{2k})$ is given by

$$1 - \frac{2k^2 M^2}{t_{2k-1} r^2} - C_{k-1}(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2(k-2)}(t_{2(k-1)-1} + t_{2(k-1)}), t_{2(k-1)-1}, \lambda_{2(k-1)}(t_{2k-1} + t_{2k}))^{-1} \frac{1}{2\lambda_{2(k-1)}}.$$

Our induction hypothesis allows us to choose $\lambda_2, \lambda_4, \dots, \lambda_{2(k-2)}$, and r sufficiently large so that $C_i > 0$ for $2 \leq i \leq k-1$. Furthermore, the first condition of the induction hypothesis implies that the constant

$$C_{k-1}(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2(k-2)}(t_{2(k-1)-1} + t_{2(k-1)}), t_{2(k-1)-1}, t_{2(k-1)})$$

does not depend on $t_{2(k-1)}$, which in turn allows us to conclude that

$$C_{k-1}(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2(k-2)}(t_{2(k-1)-1} + t_{2(k-1)}), t_{2(k-1)-1}, \lambda_{2(k-1)}(t_{2k-1} + t_{2k}))$$

does not depend on $\lambda_{2(k-1)}$. Therefore, by taking $\lambda_{2(k-1)}$ and r sufficiently large we can make it so that

$$C_k(r, t_1, \lambda_2(t_3 + t_4), \dots, \lambda_{2k-2}(t_{2k-1} + t_{2k}), t_{2k-1}, t_{2k}) > 0.$$

This establishes the second condition for C_k . It follows by induction that it is true for all $i \geq 2$.

As for the last statement, simply observe that if we are given positive numbers $t_1, t_3, \dots, t_{2i-1}, t_{2i}$ we can substitute $t_{2j} = \lambda_{2j}(t_{2j+1} + t_{2j+2})$ into the $2j$ -th position for $1 \leq j < i$. By condition (ii) we can then choose $\lambda_2, \lambda_4, \dots, \lambda_{2(i-1)}$, and r sufficiently large so that $C_i > 0$. \square

In the following proposition, we will prove the required local estimate for $D^k(u)$. We will use the definition of the C_i outlined in the hypothesis of the above lemma.

Proposition 6.10. *For $u \in \mathcal{S}_{k+1}$, given any $t_1, t_3, \dots, t_{2k-1}, t_{2k} > 0$ we can choose positive $t_2, t_4, \dots, t_{2(k-1)}, r$ sufficiently large so that $C_i > 0$ for $1 \leq i \leq k$, and so that*

$$\begin{aligned} C_k \|b_k^2 D^k(u)\|_{B_{2e(r)r}}^2 &\leq \frac{1}{2t_k} \|b_r^{k+1} D^{k+1}(u)\|_{B_{2e(r)r}}^2 \\ &+ \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \cdots \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2}\right) C_1^{-1} C_2^{-1} \cdots C_{k-2}^{-1} C_{k-1}^{-1} \|u\|_{B_{2e(r)r}}^2. \end{aligned}$$

Proof. We will prove this by induction, the $k = 2$ case being done in Proposition 6.6. So assume the Proposition is true for $k-1$. This means that for $u \in \mathcal{S}_k$, given any $t_1, t_3, \dots, t_{2(k-1)-1}, t_{2(k-1)} > 0$ we can choose positive $t_2, t_4, \dots, t_{2(k-2)}, r$ sufficiently

large so that $C_i > 0$ for $1 \leq i \leq k-1$, and so that

$$\begin{aligned} C_{k-1} \|b_r^{k-1} D^{k-1}(u)\|_{B_{2r}}^2 &\leq \frac{1}{2t_{2(k-1)}} \|b_r^k D^k(u)\|_{B_{2r}}^2 \\ &\quad + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \left(\frac{t_3}{2} + \frac{t_4}{2}\right) \cdots \\ &\quad \left(\frac{t_{2(k-1)-1}}{2} + \frac{t_{2(k-1)}}{2}\right) C_1^{-1} C_2^{-1} \cdots C_{k-3}^{-1} C_{k-2}^{-1} \|u\|_{B_{2r}}^2. \end{aligned}$$

It should be noted, as we observed in Remark 6.5, that such an estimate holds merely for $u \in C^\infty(\mathcal{V})$. When we substitute this estimate into the derived estimate for $D^k(u)$ we will use this fact without explicit mention.

By Lemma 6.9, we can choose $t_2, t_4, \dots, t_{2(k-1)}$ and r sufficiently large so that $C_i > 0$ for $1 \leq i \leq k$. Furthermore, in the proof of that lemma, we showed how we could choose $t_2, t_4, \dots, t_{2(k-1)}$ so that the C_{k-1} would not depend on $t_{2(k-1)}$. In particular, this means that for these $C_i > 0$ for $1 \leq i \leq k-1$ the above estimate for $D^{k-1}(u)$ holds with all the $C_i > 0$. Therefore, all that we need to do is show that the above estimate holds.

To obtain an estimate for $D^k(u)$ we start by estimating $\|b_r^k D^k(u)\|_{B_{2e(r)r}}^2$. We begin by fixing a smooth compactly supported function η such that $\eta = 1$ on $B_{3e(r)r}$ and η vanishing outside of $B_{4e(r)r}$. We then have that

$$\begin{aligned} \langle b_r^k D^k(\eta u), b_r^k D^k(\eta u) \rangle &= \langle b_r^{2k} D^k(\eta u), D^k(\eta u) \rangle \\ &= \langle D(b_r^{2k} D^k(\eta u)), D^{k-1}(\eta u) \rangle \\ &= \langle M_{b_r^{2k}}(D^k(\eta u)), D^{k-1}(\eta u) \rangle + \langle b_r^{2k} D^{k+1}(\eta u), D^{k-1}(\eta u) \rangle \\ &= \langle M_{b_r^{2k}}(D^k(\eta u)), D^{k-1}(\eta u) \rangle + \langle b_r^{k+1} D^{k+1}(\eta u), b_r^{k-1} D^{k-1}(\eta u) \rangle \end{aligned}$$

where the second equality follows from Proposition 6.1.

As b_r^k is supported inside $\overline{B_{2e(r)r}}$, and $\eta = 1$ on $B_{3e(r)r}$ we have

$$\begin{aligned} \langle b_r^k D^k(u), b_r^k D^k(u) \rangle_{B_{2e(r)r+\varepsilon}} &= \langle M_{b_r^{2k}}(D^k(u)), D^{k-1}(u) \rangle_{B_{2e(r)r+\varepsilon}} + \langle b_r^{k+1} D^{k+1}(u), b_r^{k-1} D^{k-1}(u) \rangle_{B_{2e(r)r+\varepsilon}} \end{aligned}$$

for any $\varepsilon > 0$ sufficiently small. In particular, this then gives

$$\begin{aligned} \langle b_r^k D^k(u), b_r^k D^k(u) \rangle_{B_{2e(r)r}} &= \langle M_{b_r^{2k}}(D^k(u)), D^{k-1}(u) \rangle_{B_{2e(r)r}} + \langle b_r^{k+1} D^{k+1}(u), b_r^{k-1} D^{k-1}(u) \rangle_{B_{2e(r)r}}. \end{aligned}$$

Using Cauchy-Schwarz, the bound (6.5), and the fact that b_r^{k-1} does not have any zeros in $B_{2e(r)r}$, we have

$$\begin{aligned}
 & \left| \langle M_{b_r^{2k}}(D^k(u)), D^{k-1}(u) \rangle_{B_{2e(r)r}} \right| \\
 &= \left| \left\langle \frac{1}{b_r^{k-1}} M_{b_r^{2k}}(D^k(u)), b_r^{k-1} D^{k-1}(u) \right\rangle_{B_{2e(r)r}} \right| \\
 &\leq \frac{2k^2 M^2}{t_{2k-1} r^2} \left\| \frac{b_r^{2k-1}}{b_r^{k-1}} D^k(u) \right\|_{B_{2e(r)r}}^2 + \frac{t_{2k-1}}{2} \|b_r^{k-1} D^{k-1}(u)\|_{B_{2e(r)r}}^2 \\
 &= \frac{2k^2 M^2}{t_{2k-1} r^2} \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 + \frac{t_{2k-1}}{2} \|b_r^{k-1} D^{k-1}(u)\|_{B_{2e(r)r}}^2.
 \end{aligned}$$

Similarly we have

$$\left| \langle b_r^{2k} D^{k+1}(u), D^{k-1}(u) \rangle_{B_{2e(r)r}} \right| \leq \frac{1}{2t_{2k}} \|b_r^{k+1} D^{k+1}(u)\|_{B_{2e(r)r}}^2 + \frac{t_{2k}}{2} \|b_r^{k-1} D^{k-1}(u)\|_{B_{2e(r)r}}^2.$$

These two bounds imply

$$\begin{aligned}
 \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 &\leq \frac{2k^2 M^2}{t_{2k-1} r^2} \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2} \right) \|b_r^{k-1} D^{k-1}(u)\|_{B_{2e(r)r}}^2 \\
 &\quad + \frac{1}{2t_{2k}} \|b_r^{k+1} D^{k+1}(u)\|_{B_{2e(r)r}}^2.
 \end{aligned}$$

We now substitute the estimate for $\|b_r^{k-1} D^{k-1}(u)\|_{B_{2e(r)r}}^2$, as given in the induction hypothesis, to obtain

$$\begin{aligned}
 \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 &\leq \frac{2k^2 M^2}{t_{2k-1} r^2} \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 \\
 &\quad + \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2} \right) C_{k-1}^{-1} \frac{1}{2t_{2(k-1)}} \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 \\
 &\quad + \left(\frac{t_1}{2} + \frac{t_2}{2} \right) \left(\frac{t_3}{2} + \frac{t_4}{2} \right) \cdots \left(\frac{t_{2(k-1)-1}}{2} + \frac{t_{2(k-1)}}{2} \right) \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2} \right) C_1^{-1} C_2^{-1} \\
 &\quad \cdots C_{k-3}^{-1} C_{k-2}^{-1} C_{k-1}^{-1} \|u\|_{B_{2r}}^2 \\
 &\quad + \frac{1}{2t_{2k}} \|b_r^{k+1} D^{k+1}(u)\|_{B_{2e(r)r}}^2.
 \end{aligned}$$

This can be re-written as

$$\begin{aligned}
 & \left(1 - \frac{2k^2 M^2}{t_{2k-1} r^2} - \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2} \right) C_{k-1}^{-1} \frac{1}{2t_{2(k-1)}} \right) \|b_r^k D^k(u)\|_{B_{2e(r)r}}^2 \\
 &\leq \frac{1}{2t_{2k}} \|b_r^{k+1} D^{k+1}(u)\|_{B_{2e(r)r}}^2 + \left(\frac{t_1}{2} + \frac{t_2}{2} \right) \left(\frac{t_3}{2} + \frac{t_4}{2} \right) \\
 &\quad \cdots \left(\frac{t_{2(k-1)-1}}{2} + \frac{t_{2(k-1)}}{2} \right) \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2} \right) C_1^{-1} C_2^{-1} \cdots C_{k-3}^{-1} C_{k-2}^{-1} C_{k-1}^{-1} \|u\|_{B_{2e(r)r}}^2
 \end{aligned}$$

By definition, the coefficient on the left hand side is precisely C_k , and so we have established the estimate for $D^k(u)$. \square

The local estimates we obtain, from the above proposition, can be promoted to a global estimate by taking $r \rightarrow \infty$. All we have to note is that we have explicit

formulas for the constants. We know that $C_1(r, t_1) = \left(1 - \frac{2M^2}{t_1 r^2}\right)$, and in general

$$C_i(r, t_1, t_2, \dots, t_{2i-1}, t_{2i}) = \left(1 - \frac{2i^2 M^2}{t_{2i-1} r^2} - \left(\frac{t_{2i-1}}{2} + \frac{t_{2i}}{2}\right) C_{i-1}^{-1} \frac{1}{2t_{2(i-1)}}\right).$$

Therefore, it follows that as $r \rightarrow \infty$ we have that $C_1 \rightarrow 1$, and in general

$$C_i \rightarrow C_i^0 := \left(1 - \left(\frac{t_{2i-1}}{2} + \frac{t_{2i}}{2}\right) (C_{i-1}^0)^{-1} \frac{1}{2t_{2(i-1)}}\right).$$

One can then prove an analogous result to Lemma 6.9 for the C_i^0 . We will not do this as the proof is exactly the same as the proof of Lemma 6.9. Using this fact, and taking $r \rightarrow \infty$ in the local estimate in Proposition 6.10, we obtain the following theorem.

Theorem 6.11. *Fix $u \in \mathcal{S}_{k+1}$, given positive $t_1, t_3, t_5, \dots, t_{2k-3}, t_{2k-1}, t_{2k}$ we can find positive $t_2, t_4, \dots, t_{2(k-2)}, t_{2(k-1)}$ so that $C_i^0 > 0$ for all $1 \leq i \leq k$, and so that*

$$\begin{aligned} C_k^0 \|D^k(u)\|^2 &\leq \frac{1}{2t_{2k}} \|D^{k+1}(u)\|^2 + \left(\frac{t_1}{2} + \frac{t_2}{2}\right) \\ &\quad \dots \left(\frac{t_{2(k-1)-1}}{2} + \frac{t_{2(k-1)}}{2}\right) \left(\frac{t_{2k-1}}{2} + \frac{t_{2k}}{2}\right) (C_1^0)^{-1} (C_2^0)^{-1} \dots (C_{k-2}^0)^{-1} (C_{k-1}^0)^{-1} \|u\|^2. \end{aligned}$$

In particular $\|D^k(u)\| < \infty$, and $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$.

With the aid of this theorem, we conclude the paper by presenting the following proof of Theorem 2.7.

Proof of Theorem 2.7. The above theorem shows us that $\mathcal{S}_k \subseteq \mathcal{S}_{k-1} \subseteq \dots \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_1$ for $1 \leq k \leq m+1$. Fixing $u, v \in \mathcal{S}_k$, we have using Theorem 6.2 that

$$\langle D^k u, v \rangle = \langle D^{k-1} u, Dv \rangle = \langle D^{k-2} u, D^2 v \rangle = \dots = \langle Du, D^{k-1} v \rangle = \langle u, D^k v \rangle.$$

This implies that D^k satisfies the negligible boundary condition (*l-Neg*) for $l = k$ and the conclusion follows from Theorem 2.4. \square

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LASHI BANDARA, MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG, SE-412 96, GOTHENBURG, SWEDEN

URL: <http://www.math.chalmers.se/~lashitha>

E-mail address: lashi.bandara@chalmers.se

HEMANTH SARATCHANDRAN, MATHEMATICAL SCIENCES, BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, NO. 5 YIHEYUAN ROAD, HAIDIAN DISTRICT, PEKING UNIVERSITY, BEIJING 100871, CHINA P.R.

E-mail address: hemanth.saratchandran@gmail.com